Exercise 13.1

Question 1: Evaluate the Given limit: $\lim_{x\to 3} x+3$

Solution 1: $\lim_{x\to 3} x+3 = 3 + 3 = 6$

Question 2: Evaluate the Given limit: $\lim_{x\to \pi}\left(x-\frac{22}{7}\right)$

Solution 2: $\lim_{x\to \pi}\left(x-\frac{22}{7}\right)=\left(\pi-\frac{22}{7}\right)$

Question 3: Evaluate the Given limit: $\lim_{r\to 1} \pi r^2$

Solution 3: $\lim_{r\to 1} \pi r^2 = \pi(1^2) = \pi$

Question 4: Evaluate the Given limit: $\lim_{x\to 1}\frac{4x+3}{x-2}$

Solution 4: $\lim_{x\to 1}\frac{4x+3}{x-2}=\frac{4(4)+3}{4-2}=\frac{16+3}{2}=\frac{19}{2}$

Question 5: Evaluate the Given limit: $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

Solution 5: $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}=\frac{(-1)^{10}+(-1)^5+1}{-1-1}=\frac{1-1+1}{-2}=-\frac{1}{2}$

Question 6: Evaluate the Given limit: $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Solution 6: $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Put x + 1 = y so that y ⟶ 1 as x ⟶0.

Accordingly, $\lim_{x\to 0}\frac{(x+1)^5-1}{x}=\lim_{x\to 1}\frac{(y)^5-1}{y-1}$

$=5.1^{5-1}$ $\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$

= 5

$\therefore \lim_{x\to 0}\frac{(x+1)^5-1}{x}=5$

Question 7: Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Solution 7: At x = 2, the value of the given rational function takes the form $\frac{0}{0}$

$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$=\lim_{x\to 2}\frac{3x+5}{x+2}$

$=\frac{3(2)+5}{2+2}$

$=\frac{11}{4}$

Question 8: Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Solution 8: At x = 2, the value of the given rational function takes the form $\frac{0}{0}$

$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$

$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{(2x+1)}$

$=\frac{(3+3)(3^2+9)}{2(3)+1}$

$= \dfrac{6 \times 18}{7}$

$= \dfrac{108}{7}$

Question 9: Evaluate the Given limit: $\lim_{x \to 0} \dfrac{ax+b}{cx+1}$.

Solution 9:

$$\lim_{x \to 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

Question 10: Evaluate the Given limit: $\lim_{z \to 1} \dfrac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Solution 10: $\lim_{z \to 1} \dfrac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

At z = 1, the value of the given function takes the form $\dfrac{0}{0}$

Put $z^{\frac{1}{6}}$ = x so that z →1 as x →1.

Accordingly, $\lim_{z \to 1} \dfrac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = \lim_{x \to 1} \dfrac{x^2-1}{x-1}$

$= \lim_{x \to 1} \dfrac{x^2-1}{x-1}$

$= 2.1^{2-1}$ $\quad \left[\lim_{x \to a} \dfrac{x^n - a^n}{x-a} = na^{n-1} \right]$

$= 2$

$$\lim_{z \to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = 2$$

Question 11: Evaluate the Given limit: $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$, a + b + c ≠ 0

Solution 11: $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a} = \frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a}$

$= \frac{a + b + c}{a + b + c}$

= 1 [a + b + c ≠ 0]

Question 12: Evaluate the given limit: $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

Solution 12: $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

At x = -2, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2} = \lim_{x \to -2} \frac{\left(\frac{2 + x}{2x}\right)}{x + 2}$

$= \lim_{x \to -2} \frac{1}{2x}$

$= \frac{1}{2(-2)} = \frac{-1}{4}$

Question 13: Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{bx}$

Solution 13: $\lim_{x \to 0} \frac{\sin ax}{bx}$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{ax}{bx}$

$$= \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\frac{a}{b}$$

$$= \frac{a}{b}\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right) \qquad [x\to 0\Rightarrow ax\to 0]$$

$$= \frac{a}{b}\times 1 \qquad \left[\lim_{x\to 0}\left(\frac{\sin y}{y}\right)\right]$$

$$= \frac{a}{b}$$

Question 14: Evaluate the given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}, a, b \neq 0$

Solution 14: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}, a, b \neq 0$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{ax}\right)\times bx}$

$$= \frac{a}{b}\times\frac{\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{bx\to 0}\left(\frac{\sin bx}{ax}\right)} \qquad \left[\begin{array}{l} x\to 0\Rightarrow ax\to 0 \\ and\ x\to 0\Rightarrow bx\to 0\end{array}\right]$$

$$= \frac{a}{b}\times\frac{1}{1} \qquad \left[\lim_{x\to 0}\left(\frac{\sin y}{y}\right)=1\right]$$

$$= \frac{a}{b}$$

Question 15: Evaluate the Given limit: $\lim_{x\to \pi}\frac{\sin(\pi - x)}{\pi(\pi - x)}$

Solution 15: $\lim_{x\to \pi}\frac{\sin(\pi - x)}{\pi(\pi - x)}$

It is seen that x → π ⇒ (π - x) → 0

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

$$=\frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{1}{\pi}$$

Question 16: Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi - x}$

Solution 16: $\lim_{x\to 0}\frac{\cos x}{\pi - x}=\frac{\cos 0}{\pi - 0}=\frac{1}{\pi}$

Question 17: Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Solution 17: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1}$ $\qquad \left[\cos x = 1-2\sin^2\frac{x}{2}\right]$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$= 4\frac{\left(\lim_{x\to 0}\frac{\sin^2 x}{x^2}\right)^2}{\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2} \qquad \left[x\to 0\Rightarrow \frac{x}{2}\to 0\right]$$

$$= 4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$= 4$

Question 18: Evaluate the given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Solution 18: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$

$$= \frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}\left(a+\cos x\right)$$

$$= \frac{1}{b}\left(\frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)}\right)\times\lim_{x\to 0}\left(a+\cos x\right)$$

$$= \frac{1}{b}\times\left(a+\cos 0\right) \qquad \left[\lim_{y\to 0}\frac{\sin x}{x}=1\right]$$

$$= \frac{a+1}{b}$$

Question 19: Evaluate the given limit: $\lim_{x\to 0} x\sec x$

Solution 19: $\lim_{x\to 0} x\sec x = \lim_{x\to 0}\frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$

Question 20: Evaluate the given limit: $\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx}$ a, b, a + b$\neq 0$

Solution 20: At x = 0, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx}$

$$= \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax + bx}{ax + bx\left(\frac{\sin bx}{bx}\right)}$$

$$= \frac{\left(\lim_{x\to 0}\frac{\sin ax}{ax}\right)\times \lim_{x\to 0}(ax) + \lim_{x\to 0}(bx)}{\lim_{x\to 0} ax + \lim_{x\to 0} bx\left(\lim_{x\to 0}\frac{\sin bx}{bx}\right)}$$ [As x $\to 0 \Rightarrow$ ax $\to 0$ and bx $\to 0$]

$$= \frac{\lim_{x\to 0}(ax) + \lim_{x\to 0} bx}{\lim_{x\to 0} ax + \lim_{x\to 0} bx}$$ $\left[\lim_{y\to 0}\frac{\sin x}{x} = 1\right]$

$$= \frac{\lim_{x\to 0}(ax + bx)}{\lim_{x\to 0}(ax + bx)}$$

$= \lim_{x\to 0}(1)$

= 1

Question 21: Evaluate the given limit: $\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$

Solution 21: At x = 0, the value of the given function takes the form $\infty - \infty$

Now, $\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1 - \cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\frac{\left(\frac{1 - \cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{0}{1} \qquad \left[\lim_{y\to 0}\frac{1-\cos x}{x}=0 \text{ and } \lim_{y\to 0}\frac{\sin x}{x}=1\right]$$

= 0

Question 22: Evaluate the given limit: $\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

Solution 22: $\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

At x = $\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$

Now, put So that x – $\frac{\pi}{2}$ = y so that x $\to$ $\frac{\pi}{2}$, y $\to$ 0

$$\therefore \lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0}\frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$= \lim_{y\to 0}\frac{\tan(\pi+2y)}{y}$$

$$= \lim_{y\to 0}\frac{\tan 2y}{y} \qquad [\tan(\pi+2y)=\tan 2y]$$

$$= \lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$$

$$= \lim_{y\to 0}\left(\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}\right)$$

$$= \left(\lim_{y\to 0}\frac{\sin 2y}{2y}\right)\times\lim_{y\to 0}\left(\times\frac{2}{\cos 2y}\right) \qquad [y\to 0 \Rightarrow 2y\to 0]$$

$$= 1\times\frac{2}{\cos 0} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$= 1\times\frac{2}{1}$$

= 2

Question 23: Find $\lim_{x\to 0}$ f(x) and $\lim_{x\to 1}$ f(x), where f(x) = $\begin{cases} 2x+3, & x\le 0 \\ 3(x+1), & x>0 \end{cases}$

Solution 23: The given function is

$$f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} [2x + 3] = 2(0) + 3 = 3$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x + 1) = 3(0 + 1) = 3$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (x + 1) = 3(1 + 1) = 6$

$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x) = 6$

Question 24: Find $\lim_{x\to 1} f(x)$, when $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x-1, & x > 1 \end{cases}$

Solution 24:

The given function is

$$f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x-1, & x > 1 \end{cases}$$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1} \left[x^2-1\right] = 1^2 - 1 = 1 - 1 = 0$

It is observed that $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question 25: Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution 25: The given function is

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[\frac{|x|}{x}\right]$

$= \lim_{x\to 0} \left(\frac{-x}{x}\right)$ [When x is negative, $|x|$ = -x]

$= \lim_{x\to 0} (-1)$

= -1

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$

$= \lim_{x\to 0}\left(\frac{x}{x}\right)$ [When x is positive, $|x|$ = x]

$= \lim_{x\to 0}(1)$

= 1

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 26: Find $\lim_{x\to 0} f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution 26: The given function is

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{x}{|x|}\right]$

$= \lim_{x\to 0}\left(\frac{x}{-x}\right)$ [When x < 0, $|x|$ = -x]

$= \lim_{x\to 0}(-1)$

= -1

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]$

$= \lim_{x\to 0}\left(\frac{x}{x}\right)$ [When x > 0, $|x|$ = x]

$= \lim_{x\to 0}(1)$

= 1

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 27: Find $\lim_{x\to 5} f(x)$, where f(x) = $|x|$ - 5

Solution 27: The given function is f(x) = $|x|$ - 5

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}(|x|-5)$

$= \lim_{x\to 5}(x-5)$ [When x > 0, $|x|$ = x]

= 5- 5

= 0

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}(|x|-5)$

$= \lim_{x\to 5}(x-5)$ [When x > 0, $|x|$ = x]

= 5 – 5

= 0

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 0$

Hence, $\lim_{x\to 5} f(x) = 0$

Question 28: Suppose f(x) = $\begin{cases} a+bx, & x<0 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$ and if $\lim_{x\to 1} f(x) =$ f(1) what are possible values of a and b?

Solution 28: The given function is

$$f(x) = \begin{cases} a+bx, & x<0 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(a+bx) = a+b$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(b-ax) = b-a$

f(1) = 4

It is given that $\lim_{x\to 1} f(x)$ = f(1).

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$ = f(1)

$\Rightarrow$ a + b = 4 and b – a = 4

On solving these two equations, we obtain a = 0 and b = 4.

Thus, the respective possible values of a and b are 0 and 4.

Question 29: Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and define a function

$f(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$

What is $\lim_{x\to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots, a_n$. Compute $\lim_{x\to a} f(x)$.

Solution 29: The given function is $f(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$

$\lim_{x\to a_1} f(x) = \lim_{x\to a_1}[(x - a_1)(x - a_2)\ldots(x - a_n)]$

$= (a_1 - a_1)(a_1 - a_2)\ldots(a_1 - a_n) = 0$

$\therefore \lim_{x\to a_1} f(x) = 0$

Now, $\lim_{x\to a} f(x) = \lim_{x\to a}[(x - a_1)(x - a_2)\ldots(x - a_n)]$

= (a – a_1) (a – a_2)….. (a – a_n)

$\therefore \lim_{x \to a} f(x)$ = (a – a_1) (a – a_2)….. (a – a_n)

Question 30: If f(x) = $\begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>1 \end{cases}$.

For what value (s) of does $\lim_{x \to a} f(x)$ exists?

Solution 30: The given function is

If f(x) = $\begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>1 \end{cases}$.

When a = 0,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(|x|+1)$

$= \lim_{x \to 0}(-x+1)$ [If x < 0, $|x|$ = –x]

= 0 + 1

= 1

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(|x|+1)$

$= \lim_{x \to 0}(x-1)$ [If x > 0, $|x|$ = –x]

= 0 – 1

= –1

Here, it is observed that $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

$\therefore \lim_{x \to 0} f(x)$ does not exist.

When a < 0,

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-}(|x|+1)$

$= \lim_{x \to a}(-x+1)$ [x < a < 0 $\Rightarrow$ $|x|$ = –x]

= –a + 1

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+}(|x|+1)$

$= \lim_{x \to a}(-x+1)$ [a < x < 0 $\Rightarrow$ $|x|$ = –x]

= –a + 1

$\therefore \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = -a+1$

Thus, limit of f(x) exists at x = a, where a < 0.
When a > 0

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x|+1)$

$= \lim_{x \to a} (-x-1)$ [$0 < x < a \Rightarrow |x| = x$]

= a – 1

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x|-1)$

$= \lim_{x \to a} (-x-1)$ [$0 < x < a \Rightarrow |x| = x$]

= a – 1

$\therefore \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = a-1$

Thus, limit of f(x) exists at x = a, where a > 0.
Thus, $\lim_{x \to a} f(x)$ exists for all a $\neq 0$.

Question 31: If the function f(x) satisfies, $\lim_{x \to 1} \frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

Solution 31: $\lim_{x \to 1} \frac{f(x)-2}{x^2-1} = \pi$

$$\Rightarrow \frac{\lim_{x \to 1}(f(x)-2)}{\lim_{x \to 1}(x^2-1)} = \pi$$

$$\Rightarrow \lim_{x \to 1}(f(x)-2) = \pi \lim_{x \to 1}(x^2-1)$$

$$\Rightarrow \lim_{x \to 1}(f(x)-2) = \pi(1^2-1)$$

$$\Rightarrow \lim_{x \to 1}(f(x)-2) = 0$$

$$\Rightarrow \lim_{x \to 1} f(x) - \lim_{x \to 1} 2 = 0$$

$$\Rightarrow \lim_{x \to 1} f(x) - 2 = 0$$

$$\therefore \lim_{x \to 1} f(x) = 2$$

Question 32: If f(x) = $\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1. \\ nx^3+m, & x>1 \end{cases}$

For what integers m and n does $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists?

Solution 32: f(x) = $\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1.. \\ nx^3+m, & x>1 \end{cases}$

$\lim_{x\to 0} f(x)=\lim_{x\to 0}(mx^2+n)$

= $m(0)^2+n$

= n

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(nx+m)$

= $n(0)+m$

= m

Thus, $\lim_{x\to 0^+} f(x)$ exists if m = n.

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(nx+m)$

= n (1) + m

= m + n

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(nx^3+m)$

= n $(1)^3$ + m

= m + n

$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)$.

Thus, $\lim_{x\to 1} f(x)$ exists for any internal value of m and n.

Exercise 13.2

Question 1: Find the derivative of $x^2 - 2$ at x = 10.

Solution 1: Let f(x) = $x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \to 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h}$$

$$= \lim_{h \to 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \to 0} (20 + h) = 20 + 0 = 20$$

Thus, the derivative of $x^2 - 2$ at x = 10 is 20.

Question 2: Find the derivative of 99x at x = 100.

Solution 2: Let f(x) = 99x. Accordingly,

$$f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \to 0} \frac{99(100+h) - 99(100)}{h}$$

$$= \lim_{h \to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h \to 0} \frac{99h}{h}$$

$$= \lim_{h \to 0} (99) = 99$$

Thus, the derivative of 99x at x = 100 is 99.

Question 3: Find the derivative of x at x = 1.

Solution 3: Let f(x) = x. Accordingly,

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h)-1}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{h\to 0}(1)=1$$

Thus, the derivative of x at x = 1 is 1.

Question 4: Find the derivative of the following functions from first principle.

(i) $x^3 - 27$

(ii) (x – 1) (x – 2)

(iii) $\frac{1}{x^2}$

(iv) $\frac{x+1}{x-1}$

Solution 4: (i) Let f(x) = x^3 – 27. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{[(x+h)^3-27]-(x^3-27)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$

$$=\lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}$$

$$=\lim_{h\to 0}(h^3+3x^2h+3xh^2)$$

$= 0 + 3x^2 + 0 = 3x^2$

(ii) Let f(x) = (x – 1) (x – 2). Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$$

$$=\lim_{h\to 0}\frac{(x^2+hx-2x+hx+h^2-2h-x-h+2)-(x^2-2x-x+2)}{h}$$

$$=\lim_{h\to 0}\frac{(hx+hx+h^2-2h-h)}{h}$$

$$=\lim_{h\to 0}\frac{2hx+h^2-3h}{h}$$

$$=\lim_{h\to 0}(2x+h-3)$$

= 2x – 3

(iii) Let f(x) = $\frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-2hx-h^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h^2-2x}{x^2(x+h)^2}\right]$$

$$=\frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}$$

(iv) Let f(x) = $\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Question 5: For the function

$$F(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^2}{2}+x+1$$

Prove that f '(1) = 100 f '(0)

Solution 5: The given function is

$$F(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\cdots+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\cdots+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+\cdots+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+\cdots+x+1$$

At x = 0,

f'(0) = 1

At x =1,

$$f'(1)=1^{99}+1^{98}+\cdots+1+1=[1+1+\cdots+1+1]_{100\text{ terms}}=1\times100=100$$

Thus, f'(1) = 100 f'(0)

Question 6: Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}x+a^n$ for some fixed real number a.

Solution 6: Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}x+a^n$

$$\frac{d}{dx}f(x)=\frac{d}{dx}(x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}x+a^n)$$

$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+\cdots+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\cdots+a^{n-1}+a^n(0)$$

$$\therefore f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\cdots+a^{n-1}$$

Question 7: For some constants a and b, find the derivative of

(i) (x – a) (x – b)

(ii) $(ax^2 + b)^2$

(iii) $\frac{x-a}{x-b}$

Solution 7: (i) Let f (x) = (x – a) (x – b)

$\Rightarrow f(x) = x^2 - (a+b)x + ab$

$\therefore f'(x) = \frac{d}{dx}(x^2 - (a+b)x + ab)$

$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$f'(x) = 2x - (a+b) + 0$

$= 2x - a - b$

(ii) Let f(x) = $(ax^2 + b)^2$

$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$

$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$

$= a^2\frac{d}{dx}(x^4) + 2ab\frac{d}{dx}(x^2) + \frac{d}{dx}b^2$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$

$= 4a^2x^3 + 4abx$

$= 4ax(ax^2 + b)$

(iii) Let f(x) = $\frac{x-a}{x-b}$

$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

$$=\frac{x-b-x+a}{(x-b)^2}$$

$$=\frac{a-b}{(x-b)^2}$$

Question 8: Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.

Solution 8: Let f(x) = $\frac{x^n-a^n}{x-a}$

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$

$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Question 9: Find the derivative of

(i) $2x-\frac{3}{4}$

(ii) $(5x^3 + 3x – 1)(x – 1)$

(iii) $x^{-3}(5 + 3x)$

(iv) $x^5(3 – 6x^{-9})$

(v) $x^{-4}(3 – 4x^{-5})$

(vi) $\frac{2}{x+1}-\frac{x^2}{3x-1}$

Solution 9: (i) Let f(x) = $2x-\frac{3}{4}$

$$f'(x)=\frac{d}{dx}\left(2x-\frac{3}{4}\right)$$

$$=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)$$

= 2 – 0

= 2

(ii) Let f (x) = $(5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$f'(x)=(5x^3+3x-1)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}(5x^3+3x-1)$$

$= (5x^3 + 3x - 1)(1) + (x - 1)(5.3x^2 + 3 - 0)$

$= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$

$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$

$= 20x^3 - 15x^2 + 6x - 4$

(iii) Let f (x) = $x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$f'(x)=x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}(x^{-3})$$

$$=x^{-3}(0+3)+(5+3x)(3x^{-3-1})$$

$$=x^{-3}(3)+(5+3x)(3x^{-4})$$

$$=3x^{-3}-15x^{-4}-9x^{-3}$$

$$=-6x^{-3}-15x^{-4}$$

$$=-3x^{-3}\left(2+\frac{5}{x}\right)$$

$$=\frac{-3x^{-3}}{x}(2x+5)$$

$$=\frac{-3}{x^4}(5+2x)$$

(iv) Let f (x) = $x^5(3-6x^{-9})$

By Leibnitz product rule,

$$f'(x)=x^5\frac{d}{dx}(3-6x^{-9})+(3-6x^{-9})\frac{d}{dx}(x^5)$$

$$=x^5\{0-6(-9)x^{-9-1}\}+(3-6x^{-9})(5x^4)$$

$$=x^5(54x^{-10})+15x^4-30x^{-5}$$

$$=54x^{-5}+15x^4-30x^{-5}$$

$$=24x^{-5}+15x^4$$

$$=15x^4+\frac{24}{x^5}$$

(v) Let f (x) = $x^{-4}(3-4x^{-5})$

By Leibnitz product rule,

$$f'(x)=x^{-4}\frac{d}{dx}(3-4x^{-5})+(3-4x^{-5})\frac{d}{dx}(x^{-4})$$

$$=x^{-4}\{0-4(-5)x^{-5-1}\}+(3-4x^{-5})(-4)x^{-4-1}$$

$$=x^{-4}(20x^{-6})+(3-4x^{-5})(-4x^{-5})$$

$$=20x^{-10}-12x^{-5}+16x^{-10}$$

$$=36x^{-10}-12x^{-5}$$

$$=\frac{12}{x^{-5}}+\frac{36}{x^{10}}$$

(vi) Let f (x) = $\frac{2}{x+1}-\frac{x^2}{3x-1}$

$$f'(x)=\frac{d}{dx}\left(\frac{2}{x+1}\right)-\frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x)=\left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right]-\left[\frac{(3x-1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$=\left[\frac{(x+1)(0)-2(0)}{(x+1)^2}\right]-\left[\frac{(3x-1)(2x)-x^2(3)}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$$

Question 10: Find the derivative of cos x from first principle.

Solution 10: Let f(x) = cos x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos(x+h)-\cos(x)}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}\frac{-\sin x\sin h}{h}\right]$$

$$=-\cos x\left[\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)\right]-\sin x\left[\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=-\cos x(0)-\sin x(1) \qquad \left[\lim_{h\to 0}\frac{1-\cos h}{h}=0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$\therefore f'(x)=-\sin x$$

Question 11: Find the derivative of the following functions:

(i) sin x cos x

(ii) sec x

(iii) 5sec x + 4cos x

(iv) cosec x

(v) 3cot x + 5cosec x

(vi) 5sin x – 6cos x + 7

(vii) 2tan x – 7sec x

Solution 11: (i) Let f (x) = sin x cos x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h}$$

$$=\lim_{h\to 0}\frac{1}{2h}[2\sin(x+h)\cos(x+h)-2\sin x\cos x]$$

$$=\lim_{h\to 0}\frac{1}{2h}[\sin 2(x+h)-\sin 2x]$$

$$=\lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right]$$

$$=\lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{4x+2h}{2}\cdot\sin\frac{2h}{2}\right]$$

$$=\lim_{h\to 0}\frac{1}{2h}[\cos(2x+h)\sin h]$$

$$=\lim_{h\to 0}\cos(2x+h)\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$=\cos(2x+h)\cdot 1$$

$$=\cos 2x$$

(ii) Let f (x) = sec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\cdot\lim_{h\to 0}\frac{1}{2h}\frac{\left[-2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$

$$=\frac{1}{\cos x}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

$$=\frac{1}{\cos x}\cdot 1\cdot\frac{\sin x}{\cos x}$$

$$=\sec x\tan x$$

(iii) Let f (x) = 5sec x + 4cos x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$=\frac{5}{\cos x}\cdot\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos x)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=\frac{5}{\cos x}\cdot\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}+4[-\cos x(0)-\sin x(1)]$$

$$=\frac{5}{\cos x}\cdot\left[\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}\right]-4\sin x$$

$$=\frac{5}{\cos x}\cdot\frac{\sin x}{\cos x}\cdot 1-4\sin x$$

$$=5\sec x\tan x-4\sin x$$

(iv) Let f (x) = cosec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[\mathrm{cosec}(x+h)-\mathrm{cosec}\,x]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{\left[-\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\sin x \sin(x+h)}$$

$$= \lim_{h \to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x \sin(x+h)}\right)\cdot \lim_{\frac{h}{2} \to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x}\right)\cdot 1$$

$$= -\operatorname{cosec} x \cot x$$

(v) Let f (x) = 3cot x + 5cosec x. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0}\frac{1}{h}[3\cot(x+h) + 5\operatorname{cosec}(x+h) - 3\cot x - 5\operatorname{cosec} x]$$

$$= 3\lim_{h \to 0}\frac{1}{h}[\cot(x+h)-\cot x] + 5\lim_{h \to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec} x] \qquad \ldots(1)$$

Now, $\lim_{h \to 0}\frac{1}{h}[\cot(x+h)-\cot x] = \lim_{h \to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\left[\frac{1}{\sin x\sin(x+h)}\right]$$

$$=-1\cdot\frac{1}{\sin x\sin(x+h)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \qquad ....(2)$$

$$\lim_{h\to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec}x]=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin x\sin(x+h)}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x\sin(x+h)}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right)\cdot 1$$

$$=-\operatorname{cosec}x\cot x \qquad .....(3)$$

From (1), (2), and (3), we obtain

$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$

(vi) Let f (x) = 5sin x – 6cos x + 7. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7]$$

$$=5\lim_{h\to 0}\frac{1}{h}[\sin(x+h)-\sin x]-6\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]-6\left[-\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\right]$$

$$=5\cos x\cdot 1-6[(-\cos x)\cdot(0)-\sin x\cdot 1]$$

$$=5\cos x+6\sin x$$

(vii) Let f (x) = 2 tan x – 7 sec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x]$$

$$=2\lim_{h\to 0}\frac{1}{h}[\tan(x+h)-\tan x]-7\lim_{h\to 0}\frac{1}{h}[\sec(x+h)-\sec x]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\operatorname{cosec}(x+h)}-\frac{1}{\operatorname{cosec} x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x\sin(x+h)-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x + h - x}{\cos x\cos(x+h)}\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$= 2\left[\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right] - 7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$= 2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left[\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right] - 7\left(\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$= 2\cdot 1\cdot 1\frac{1}{\cos x\cos x} - 7\cdot 1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$= 2\sec^2 x - 7\sec x\tan x$$

Vashu Panwar

Miscellaneous Exercise

Question 1: Find the derivative of the following functions from first principle:

(i) –x

(ii) $(-x)^{-1}$

(iii) sin (x + 1)

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Solution 1: (i) Let f(x) = –x. Accordingly, f(x + h) = –(x + h)

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{-(x+h)-(-x)}{h}$$

$$= \lim_{h\to 0}\frac{-x-h+x}{h}$$

$$= \lim_{h\to 0}\frac{-h}{h}$$

$$= \lim_{h\to 0}(-1) = -1$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, f(x + h) = $\frac{-1}{(x+h)}$

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{(x+h)} - \left(\frac{-1}{x}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{x(x+h)}$$

$$= \frac{1}{x.x} = \frac{1}{x^2}$$

(iii) Let f(x) = sin(x + 1). Accordingly, f(x + h) = sin(x + h + 1)

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}[\sin(x+h+1) - \sin(x+1)]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\cos\left(\frac{2x+h+2}{2}\right)\cdot \lim_{\frac{h}{2}\to 0}\frac{1}{h}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1 \qquad \left[\lim_{h\to 0}\frac{\sin x}{x}=1\right]$$

$$= \cos(x+1)$$

(iv) Let f(x) = $\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, f(x + h) = $\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right]\cdot \lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right)\cdot 1$$

$$= -\sin\left(x - \frac{\pi}{8}\right)$$

Question 2: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + a)

Solution 2: Let f(x) = x + a. Accordingly, f(x + h) = x + h + a

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x+h+a-x-a}{h}$$

$$= \lim_{h \to 0} \left(\frac{h}{h}\right)$$

$$= \lim_{h \to 0} (1)$$

= 1

Question 3: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non- zero constants and m and n are integers): (px + q)$\left(\frac{r}{x} + s\right)$

Solution 3: Let f(x) = (px + q)$\left(\frac{r}{x} + s\right)$

By Leibnitz product rule,

$$f'(x) = (px+q)\left(\frac{r}{x} + s\right)' + \left(\frac{r}{x} + s\right)(px+q)'$$

$$= (px+q)\left(rx^{-1} + s\right)' + \left(\frac{r}{x} + s\right)(p)$$

$$= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x} + s\right)p$$

$$= (px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$$

$$= \frac{-px}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$$

$$= ps-\frac{qr}{x^2}$$

Question 4: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)(cx + d)^2$

Solution 4: Let $f'(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$$f'(x)=(ax+b)\frac{d}{dx}(cx+d)^2\frac{d}{dx}(ax+b)$$

$$= (ax+b)\frac{d}{dx}(c^2x^2+2cdx^2)+(cx+d)^2\frac{d}{dx}(ax+b)$$

$$= (ax+b)\left[\frac{d}{dx}(c^2x^2)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]$$

$$= (ax+b)(2c^2x+2cd)+(cx+d)^2a$$

$$= 2c(ax+b)(cx+d)+a(cx+d)^2$$

Question 5: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{ax+b}{cx+d}$

Solution 5: Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$= \frac{(cx+d)(a)-(ax+d)(c)}{(cx+d)^2}$$

$$= \frac{acx + ad - acx - bc}{(cx+d)^2}$$

$$= \frac{ad - bc}{(cx+d)^2}$$

Question 6: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

Solution 6: Let $f(x) = \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}$, where x $\neq$ 0

By quotient rule,

$$f^{'}(x) = \frac{(x-1)\frac{d}{dx}(x-1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0,1$$

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, x \neq 0,1$$

$$= \frac{x-1-x-1}{(x-1)^2}, x \neq 0,1$$

$$= \frac{-2}{(x-1)^2}, x \neq 0,1$$

Question 7: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2+bx+c}$

Solution 7: Let $f(x) = \frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{(ax^2+bx+c)\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$=\frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 8: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Solution 8: Let $f(x)=\frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x)=\frac{(px^2+qx+r)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

$$=\frac{(px^2+qx+r)(a)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

$$=\frac{apx^2+aqx+ar-aqx+2npx+bq}{(px^2+qx+r)^2}$$

$$=\frac{-apx^2+2bpx+ar-bq}{(px^2+qx+r)^2}$$

Question 9: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2+qx+r}{ax+b}$

Solution 9: Let $f(x)=\frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-aqx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Solution 10: Let $f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{a}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x) \qquad \left[\frac{d}{dx}\left(x^n\right)=nx^{n-1} \text{ and } \frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Question 11: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $4\sqrt{x}-2$

Solution 11: Let f(x) = $4\sqrt{x}-2$

$$f'(x)=\frac{d}{dx}(4\sqrt{x}-2)=\frac{d}{dx}(4\sqrt{x})-\frac{d}{dx}(2)$$

$$=4\frac{d}{dx}(x^{\frac{1}{2}})-0=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{-\frac{1}{2}}\right)=\frac{2}{\sqrt{x}}$$

Question 12: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n$

Solution 12: Let $f(x) = (ax + b)^n$. Accordingly, $f(x + h) = \{a(x + h) + b\}^n = (ax + ah + b)^n$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(ax+ah+b)-(ax+b)^n}{h}$$

$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{2}\left(\frac{ah}{ax+b}\right)^2+\cdots\right\}-1\right] \quad \text{(using binomial theorem)}$$

$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\cdots(\text{Terms containing higher degrees of h})\right]$$

$$=(ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\cdots\right]$$

$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$=na\frac{(ax+b)^n}{ax+b}$$

$$=na(ax+b)^{n-1}$$

Question 13: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n (cx + d)^m$

Solution 13: Let $f(x) = (ax + b)^n (cx + d)^m$

By Leibnitz product rule,

$$f'(x) = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n \qquad \ldots (1)$$

Now let $f_1(x) = (cx + d)^m$

$f_1(x + h) = (cx + ch + d)^m$

$$f_1'(x) = \lim_{h \to 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \to 0} \frac{(cx+ch+d)^m - (cx+d)^m}{h}$$

$$= (cx+d)^m \lim_{h \to 0} \frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m - 1\right]$$

$$= (cx+d)^m \lim_{h \to 0} \frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)} + \frac{m(m-1)}{2}\frac{c^2h^2}{(cx+d)^2} + \cdots\right)^m - 1\right]$$

$$= (cx+d)^m \lim_{h \to 0} \frac{1}{h}\left[\frac{mch}{(cx+d)} + \frac{m(m-1)c^2h^2}{2(cx+d)^2} + \cdots \text{(Terms containing higher degree oh h)}\right]$$

$$= (cx+d)^m \lim_{h \to 0}\left[\frac{mc}{(cx+d)} + \frac{m(m-1)c^2h^2}{2(cx+d)^2} + \cdots\right]$$

$$= (cx+d)^m \left[\frac{mch}{(cx+d)} + 0\right]$$

$$= \frac{mc(cx+d)^m}{(cx+d)}$$

$$= mc(cx+d)^{m-1}$$

$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1} \qquad \ldots (2)$$

Similarly, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$ ... (3)

Therefore, from (1), (2), and (3), we obtain

$$f'(x) = (ax+b)^n\{mc(cx+d)^{m-1}\} + (cx+d)^m\{na(ax+b)^{n-1}\}$$

$$= (ax+b)^{n-1}(cx+d)^{m-1}[mc(ax+b)+na(cx+d)]$$

Question 14: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): sin (x + a)

Solution 14: Let, f(x) = sin(x + a)

f(x + h) = sin(x + h + a)

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]\right]$$

$$= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right] \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= \cos\left(\frac{2x+2a}{2}\right)\times 1 \qquad \left[\lim_{h\to 0}\frac{\sin x}{x}=1\right]$$

$$= \cos(x+a)$$

Question 15: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): cosec x cot x

Solution 15: Let f(x) = cosec x cot x

By Leibnitz product rule,

f'(x) = cosec x(cot x)'+cot x(cosec x)' ….(1)

Let $f_1(x)$ = cot x. Accordingly, $f_1(x + h)$ = cot (x + h)

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cot(x+h)-\cot(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos(x)}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x+h)}{\sin x\sin(x+h)}\right)$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$=\frac{-1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$$=\frac{-1}{\sin x}\cdot 1\cdot\left(\lim_{h\to 0}\frac{1}{\sin(x+0)}\right)$$

$$=\frac{-1}{\sin^2 x}$$

$$=-\operatorname{cosec}^2 x$$

$$\therefore (\cot x)'=-\operatorname{cosec}^2 x \qquad \ldots (2)$$

Now, let $f_2(x)$ = cosec x. Accordingly, $f_2(x + h)$ = cosec(x + h)

By first principle,

$$f_2'(x) = \lim_{h \to 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}[\operatorname{cosec}(x+h) - \operatorname{cosec}(x)]$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right)$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x(-\operatorname{cosec}^2 x) + \cot x(-\operatorname{cosec} x \cot x)$$

$$= -\text{cosec}^3 x - \cot^2 x \text{cosec}\, x$$

Question 16: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1+\sin x}$

Solution 16: Let f(x) = $\frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x) = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1-\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{(1+\sin x)^2}$$

Question 17: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{\sin x + \cos x}{\sin x - \cos x}$

Solution 17: Let f (x) = $\frac{\sin x + \cos x}{\sin x - \cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x+\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x+\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x+\cos x)^2}$$

$$=\frac{-[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x]}{(\sin x+\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Question 18: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sec x-1}{\sec x+1}$

Solution 18: Let f(x) = $\frac{\sec x-1}{\sec x+1}$

$$f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2} = \frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$= \frac{2\sin x \sec^2 x}{(\sec x+1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x+1)^2}$$

Question 19: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin^n x$

Solution 19: Let $y = \sin^n x$.

Accordingly, for n = 1, y = sin x.

$$\therefore \frac{dy}{dx} = \cos x \text{, i.e., } \frac{d}{dx}\sin x = \cos x$$

For n = 2, $y = \sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$

= (sin x)'(sin x + sin x(sin x)' [By Leibnitz product rule]

= cos x sin x + sin x cos x

= 2sin x cos x …. (1)

For n = 3, $y = \sin^3 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$$

= $(\sin x)'\sin^2 x + \sin x(\sin x)'$ [By Leibnitz product rule]

= $\cos x \sin^2 x + \sin x(2\sin x \cos x)$ [Using (1)]

= $\cos x \sin^2 x + \sin^2 x \cos x$

$= 3\sin^2 x \cos x$

We assert that $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x\cos x$

Let our assertion be true for n = k.

i.e., $\frac{d}{dx}(\sin^k x) = k\sin^{(k-1)} x\cos x$ …. (2)

Consider

$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^{(k)} x)$

$= (\sin x)' \sin^k x + \sin x (\sin^k x)'$ [By Leibnitz product rule]

$= \cos x \sin^k x + \sin x (k\sin^{k-1} \cos x)$ [Using (2)]

$= \cos x \sin^k x + 2 \sin^k x \cos x$

$= (k + 1) \sin^k x \cos x$

Thus, our assertion is true for n = k + 1.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x\cos x$

Question 20: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a+b\sin x}{c+d\cos x}$

Solution 20: Let f(x) = $\frac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x) = \frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x) - (a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$= \frac{(c+d\cos x)(b\cos x) - (a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$= \frac{cb\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd(\cos^2 x + \sin^2 x)}{(c + d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$$

Question 21: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sin(x+a)}{\cos x}$

Solution 21: Let f(x) = $\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x) = \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a)\frac{d}{dx}(-\sin x)}{\cos^2 x} \quad \ldots \text{(i)}$$

Let g(x) = sin(x + a). Accordingly, g(x + h) = sin (x + h + a)

By first principle,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}[\sin(x+h+a) - \sin(x+a)]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h \to 0} \left[\cos\left(\frac{2x+2a+h}{h}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{h}\right)\cdot\lim_{h\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=\left(\cos\frac{2x+2a}{2}\right)\times 1 \qquad \left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=\cos(x+a) \qquad \ldots \text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{\cos x\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$=\frac{\cos(x+a-x)}{\cos^2 x}$$

$$=\frac{\cos a}{\cos^2 x}$$

Question 22: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): x^4 (5 sin x – 3 cos x)

Solution 22: Let f (x) = x^4 (5 sin x – 3 cos x)

By product rule,

$$f'(x)=x^4\frac{d}{dx}(5\sin x-3\cos x)+(5\sin x-3\cos x)\frac{d}{dx}(x^4)$$

$$=x^4\left[5\frac{d}{dx}(\sin x)-3\frac{d}{dx}(\cos x)\right]+(5\sin x-3\cos x)\frac{d}{dx}(x^4)$$

$$=x^4\left[5\cos x-3(-\sin x)\right]+(5\sin x-3\cos x)(4x^3)$$

$$=x^3[5x\cos x+3x\sin x+20\sin x-12\cos x]$$

Question 23: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x^2 + 1) cos x

Solution 23: Let f(x) = $(x^2 + 1)\cos x$

By product rule,

$$f'(x) = (x^2+1)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2+1)$$

$$= (x^2+1)(-\sin x) + \cos x(2x)$$

$$= -x^2\sin x - \sin x + 2x\cos x$$

Question 24: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax^2 + \sin x)(p + q\cos x)$

Solution 24: Let f(x) = $(ax^2 + \sin x)(p + q\cos x)$

By product rule,

$$f'(x) = (ax^2+\sin x)\frac{d}{dx}(p+q\cos x) + (p+q\cos x)\frac{d}{dx}(ax^2+\sin x)$$

$$= (ax^2+\sin x)(-q\sin x) + (p+q\cos x)(2ax+\cos x)$$

$$= -q\sin x(ax^2+\sin x) + (p+q\cos x)(2ax+\cos x)$$

Question 25: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + \cos x)(x - \tan x)$

Solution 25: Let f(x) = $(x + \cos x)(x - \tan x)$

By product rule,

$$f'(x) = (x+\cos x)\frac{d}{dx}(x-\tan x) + (x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$= (x+\cos x)\left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right] + (x-\tan x)(1-\sin x)$$

$$= (x+\cos x)\left[1 - \frac{d}{dx}(\tan x)\right] + (x-\tan x)(1-\sin x) \qquad \ldots \text{(i)}$$

Let g(x) = tan x. Accordingly, g(x + h) = tan(x + h)

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

$$=\frac{1}{\cos x}\cdot 1\cdot\left(\frac{1}{\cos(x+0)}\right)$$

$$=\frac{1}{\cos^2 x}$$

$$=\sec^2 x \qquad \ldots\text{(ii)}$$

Therefore, from (i) and (ii) ,We obtain

$$f'(x)=(x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x)$$

$$=(x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x)$$

$$=-\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)$$

Question 26: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Solution 26: Let f(x) = $\frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}(x)+7\frac{d}{dx}(\cos x)\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)[4x+5\cos x]-(4x+5\sin x)[3-7\sin x]}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28x\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

Solution 27: Let f(x) = $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$= \cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x(2x) - x^2(\cos x)}{\sin^2 x}\right]$$

$$= \frac{x\cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

Question 28: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1+\tan x}$

Solution 28: Let f(x) = $\frac{x}{1+\tan x}$

$$f^{'}(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - (x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \ f^{'}(x) = \frac{(1+\tan x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \ldots\text{(i)}$$

Let g(x) = 1 + tan x. .Accordingly, g(x + h) = 1 + tan(x+h).

By first principle,

$$g^{'}(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{1+\tan(x+h) - 1 - \tan(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x\cos x(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sinh}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h\to 0}\frac{\sinh}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1\times\frac{1}{\cos^2} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1+\tan^2 x) = \sec^2 x \qquad \ldots\text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$$

Question 29: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): (x + sec x) (x – tan x)

Solution 29: Let f(x) = (x + sec x) (x – tan x)

By product rule,

$$f(x) = (x+\sec x)\frac{d}{dx}(x-\tan x) + (x-\tan x)\frac{d}{dx}(x+\sec x)$$

$$= (x+\sec x)\left[\frac{d}{dx}(x) - \frac{d}{dx}\tan x\right] + (x-\tan x)\left[\frac{d}{dx}(x) - \frac{d}{dx}\sec x\right]$$

$$= f(x+\sec x)\left[1-\frac{d}{dx}\tan x)\right] + (x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \ldots\text{(i)}$$

Let f_1 (x) = tan x, f_2 (x) = sec x

Accordingly, f_1 (x + h)-tan(x + h) and f_2 (x + h) = sec (x + h)

$$f_1'(x) = \lim_{h\to 0}\left(\frac{f_1(x+h) - f_1(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\tan(x+h) - \tan(x)}{h}\right]$$

$$= \lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos x(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to0}\frac{1}{h}\left[\frac{\sinh}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h\to0}\frac{\sinh}{h}\right).\left(\lim_{h\to0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1\times\frac{1}{\cos^2}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1+\tan^2 x)=\sec^2 x \qquad \ldots \text{(ii)}$$

$$f'_2(x)=\lim_{h\to0}\left(\frac{f_2+(x+h)-f_2(x)}{h}\right)$$

$$=\lim_{h\to0}\left(\frac{\sec(x+h)-\sec(x)}{h}\right)$$

$$=\lim_{h\to0}\frac{1}{h}\left(\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right)$$

$$=\lim_{h\to0}\frac{1}{h}\left(\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right)$$

$$=\frac{1}{\cos x}\lim_{h\to0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{\cos x} \lim_{h\to 0} \frac{1}{h} \left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\cos(x+h)} \right]$$

$$= \sec x \frac{\left\{\lim_{h\to 0} \sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\lim_{h\to 0} \cos(x+h)}$$

$$= \sec x \cdot \frac{\sin x \cdot 1}{\cos x}$$

$$\Rightarrow \frac{d}{dx} \sec x = \sec x \tan x$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Solution 30: Let f(x) = $\frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

$$= \frac{\sin^n x.1 - x(n \sin^{n-1} x \cos x)}{\sin^{2n} x}$$

$$= \frac{\sin^{n-1} x(\sin x - nx \cos x)}{\sin^{2n} x}$$

$$= \frac{\sin x - nx \cos x}{\sin^{n+1} x}$$
