Exercise 9.1

Question 1:

Write the first five terms of the sequences whose n^{th} term is $a_n = n(n+2)$.

Solution 1:

$a_n = n(n+2)$

Substituting $n = 1, 2, 3, 4$ and 5, we obtain

$a_1 = 1(1+2) = 3$

$a_2 = 2(2+2) = 8$

$a_3 = 3(3+2) = 15$

$a_4 = 4(4+2) = 24$

$a_5 = 5(5+2) = 35$

Therefore, the required terms are 3, 8, 15, 24 and 35.

Question 2:

Write the first five terms of the sequences whose n^{th} term is $a_n = \frac{n}{n+1}$

Solution 2:

$a_n = \frac{n}{n+1}$

Substituting $n = 1, 2, 3, 4, 5$, we obtain

$a_1 = \frac{1}{1+1} = \frac{1}{2},\ a_2 = \frac{2}{2+1} = \frac{2}{3}, a_3 = \frac{3}{3+1} = \frac{3}{4},\ a_4 = \frac{4}{4+1} = \frac{4}{5},\ a_5 = \frac{5}{5+1} = \frac{5}{6}$

Therefore, the required terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ and $\frac{5}{6}$

Question 3:

Write the first five terms of the sequences whose n^{th} term is $a_n = 2^n$

Solution 3:

$a_n = 2^n$

Substituting $n = 1, 2, 3, 4, 5$, we obtain

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

Therefore, the required terms are 2, 4, 8, 16 and 32.

Question 4:

Write the first five terms of the sequences whose n^{th} term is $a_n = \frac{2n-3}{6}$

Solution 4:

Substituting $n = 1, 2, 3, 4, 5$, we obtain

$$a_1 = \frac{2\times1-3}{6} = \frac{-1}{6}$$

$$a_2 = \frac{2\times2-3}{6} = \frac{1}{6}$$

$$a_3 = \frac{2\times3-3}{6} = \frac{3}{6} = \frac{1}{2}$$

$$a_4 = \frac{2\times4-3}{6} = \frac{5}{6}$$

$$a_5 = \frac{2\times5-3}{6} = \frac{7}{6}$$

Therefore, the required terms are $\frac{-1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$.

Question 5:

Write the first five terms of the sequences whose n^{th} term is $a_n = (-1)^{n-1} 5^{n+1}$

Solution 5:

Substituting $n = 1, 2, 3, 4, 5$, we obtain

$$a_1 = (-1)^{1-1} 5^{1+1} = 5^2 = 25$$

$$a_2 = (-1)^{2-1} 5^{2+1} = -5^3 = -125$$

$$a_3 = (-1)^{3-1} 5^{3+1} = 5^4 = 625$$

$$a_4 = (-1)^{4-1} 5^{4+1} = -5^5 = -3125$$

$$a^5 = (-1)^{5-1} 5^{5+1} = 5^6 = 15625$$

Therefore, the required terms are 25, $-125, 625, -3125$ and 15625.

Question 6:

Write the first five terms of the sequences whose n^{th} term is $a_n = n\frac{n^2+5}{4}$

Solution 6:

Substituting $n = 1, 2, 3, 4, 5$, we obtain

$$a_1 = 1 \cdot \frac{1^2+5}{4} = \frac{6}{4} = \frac{3}{2}$$

$$a_2 = 2 \cdot \frac{2^2+5}{4} = 2.\frac{9}{4} = \frac{9}{2}$$

$$a_3 = 3 \cdot \frac{3^2+5}{4} = 3 \cdot \frac{14}{4} = \frac{21}{2}$$

$$a_4 = 4 \cdot \frac{4^2+5}{4} = 21$$

$$a_5 = 5 \cdot \frac{5^2+5}{4} = 5 \cdot \frac{30}{4} = \frac{75}{2}$$

Therefore, the required terms are $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}, 21$ and $\frac{75}{2}$.

Question 7:

Find the 17th and 24th term in the following sequence whose n^{th} term is $a_n = 4n - 3$

Solution 7:

Substituting $n = 17$, we obtain

$$a_{17} = 4(17) - 3 = 68 - 3 = 65$$

Substituting $n = 24$, we obtain

$$a_{24} = 4(24) - 3 = 96 - 3 = 93.$$

Question 8:

Find the 7th term in the following sequence whose n^{th} term is $a_n = \frac{n^2}{2^n}$

Solution 8:

Substituting $n = 7$, we obtain

$$a_7 = \frac{7^2}{2^7} = \frac{49}{128}$$

Question 9:

Find the 9th term in the following sequence whose n^{th} term is $a_n = (-1)^{n-1} n^3$

Solution 9:

Substituting $n = 7$, we obtain

$$a_9 = (-1)^{9-1}(9)^3 = (9)^3 = 729$$

Question 10:

Find the 20th term in the following sequence whose n^{th} term is $a_n = \frac{n(n-2)}{n+3}$

Solution 10:

Substituting $n = 20$, we obtain

$$a_{20} = \frac{20(20-2)}{20+3} = \frac{20(18)}{23} = \frac{360}{23}$$

Question 11:

Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = 3,\ a_n = 3a_{n-1} + 2$ for all $n > 1$

Solution 11:

$a_1 = 3,\ a_n = 3a_{n-1} + 2$ for $n > 1$

$$\Rightarrow a_2 = 3a_1 + 2 = 3(3) + 2 = 11$$

$$a_3 = 3a_2 + 2 = 3(11) + 2 = 35$$

$$a_4 = 3a_3 + 2 = 3(35) + 2 = 107$$

$$a_5 = 3a_4 + 2 = 3(107) + 2 = 323$$

Hence, the first five terms of the sequence are 3, 11, 35, 107 and 323.

The corresponding series is $3 + 11 + 35 + 107 + 323 + \ldots.$

Question 12:

Write the first five terms of the following sequence and obtain the corresponding series:

$$a_1 = -1,\ a_n = \frac{a_{n-1}}{n},\ n \ge 2$$

Solution 12:

$$a_1 = -1,\ a_n = \frac{a_{n-1}}{n},\ n \ge 2$$

$\Rightarrow a_2 = \frac{a_1}{2} = \frac{-1}{2}$

$a_3 = \frac{a_2}{3} = \frac{-1}{6}$

$a_4 = \frac{a_3}{4} = \frac{-1}{24}$

$a_5 = \frac{a_4}{5} = \frac{-1}{120}$

Hence, the first five terms of the sequence are $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}$ and $\frac{-1}{120}$.

The corresponding series is $(-1)+\left(\frac{-1}{2}\right)+\left(\frac{-1}{6}\right)+\left(\frac{-1}{24}\right)+\left(\frac{-1}{120}\right)+\ldots$

Question 13:

Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = a_2 = 2,\ a_n = a_{n-1} - 1,\ n > 2$

Solution 13:

$a_1 = a_2 = 2,\ a_n = a_{n-1} - 1,\ n > 2$

$\Rightarrow a_3 = a_2 - 1 = 2 - 1 = 1$

$a_4 = a_3 - 1 = 1 - 1 = 0$

$a_5 = a_4 - 1 = 0 - 1 = -1$

Hence, the first five terms of the sequence are 2, 2, 1, 0 and -1.

The corresponding series is $2+2+1+0(-1)+\ldots.$

Question 14:

The Fibonacci sequence is defined by $1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2},\ n > 2$

Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

Solution 14:

$1 = a_1 = a_2$

$a_n = a_{n-1} + a_{n-2},\ n > 2$

$\therefore a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

$\therefore$ For $n=1$, $\frac{a_{n+1}}{a_n}=\frac{a_2}{a_1}=\frac{1}{1}=1$

For $n=2$, $\frac{a_{n+1}}{a_n}=\frac{a_3}{a_2}=\frac{2}{1}=2$

For $n=3$, $\frac{a_{n+1}}{a_n}=\frac{a_4}{a_3}=\frac{3}{2}$

For $n=4$, $\frac{a_{n+1}}{a_n}=\frac{a_5}{a_4}=\frac{5}{3}$

For $n=5$, $\frac{a_{n+1}}{a_n}=\frac{a_6}{a_5}=\frac{8}{5}$.

Exercise 9.2

Question 1:
Find the sum of odd integers from 1 to 2001.

Solution 1:
The odd integers from 1 to 2001 are 1, 3, 5 …… 1999, 2001.
This sequence forms an A.P.
Here, first term, $a=1$
Common difference, $d=2$
Here, $a+(n-1)d=2001$

$\Rightarrow 1+(n-1)(2)=2001$

$\Rightarrow 2n-2=2000$

$\Rightarrow n=1001$

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$\therefore S_n=\frac{1001}{2}\left[2\times1+(1001-1)\times2\right]$

$=\frac{1001}{2}\left[2+1000\times2\right]$

$=1001\times1001$

$=1002001$

Thus, the sum of odd numbers from 1 to 2001 is 1002001.

Question 2:
Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

Solution 2:

The natural numbers lying between 100 and 1000, which are multiples of 5, are 105, 110, …. 995.

Here, $a = 105$ and $d = 5$

$a + (n-1)d = 995$

$\Rightarrow 105 + (n-1)5 = 995$

$\Rightarrow (n-1)5 = 995 - 105 = 890$

$\Rightarrow n - 1 = 178$

$\Rightarrow n = 179$

$\therefore S_n = \frac{179}{2}\left[2(105) + (179-1)(5)\right]$

$= \frac{179}{2}\left[2(105) + (178)(5)\right]$

$= 179\left[105 + (89)5\right]$

$= 179(105 + 445)$

$= (179)(550)$

$= 98450$

Thus, the sum of all natural numbers lying between 100 and 1000, which are multiples of 5, 98450.

Question 3:

In an A.P, the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20th term is -112.

Solution 3:

First term = 2

Let d be the common different of the A.P.

Therefore, the A.P. is $2,\ 2+d,\ 2+2d,\ 2+3d...$

Sum of first five terms $= 10 + 10d$

Sum of next five terms $= 10 + 35d$

According to the given condition,

$10 + 10d = \frac{1}{4}(10 + 35d)$

$\Rightarrow 40 + 40d = 10 + 35d$

$\Rightarrow 30 = -5d$

$\Rightarrow d = -6$

$\therefore a_{20} = a + (20-1)d = 2 + (19)(-6) = 2 - 114 = -112$

Thus, the 20th of the A.P. is -112.

Question 4:

How many terms of the A.P. $-6, -\frac{11}{2}, -5$, ….. are needed to give the sum -25?

Solution 4:

Let the sum of n terms of the given A.P. be -25.

It is known that,

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Where n =number of terms, $a =$ first term, and $d =$ common difference

Here, $a = -6$

$$d = -\frac{11}{2} + 6 = \frac{-11+12}{2} = \frac{1}{2}$$

Therefore, we obtain

$$-25 = \frac{n}{2}\left[2\times(-6) + (n-1)\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow -50 = n\left[-12 + \frac{n}{2} - \frac{1}{2}\right]$$

$$\Rightarrow -50 = n\left[-\frac{25}{2} + \frac{n}{2}\right]$$

$$\Rightarrow -100 = n(-25 + n)$$

$$\Rightarrow n^2 - 25n + 100 = 0$$

$$\Rightarrow n^2 - 5n - 20n + 100 = 0$$

$$\Rightarrow n(n-5) - 20(n-5) = 0$$

$$\Rightarrow n = 20 \text{ or } 5$$

Question 5:

In an A.P., if p^{th} term is 1/q and q^{th} term is 1/p, prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$.

Solution 5:

It is known that the general term of an A.P. is $a_n = a + (n-1)d$

$\therefore$ According to the given information,

$$p^{th} \text{ term} = a_p = a + (p-1)d = \frac{1}{q} \qquad \text{......}(1)$$

$$q^{th} \text{ term} = a_q = a + (q-1)d = \frac{1}{p} \qquad \text{......}(2)$$

Subtracting (2) from (1), we obtain

$$(p-1)d-(q-1)d=\frac{1}{q}-\frac{1}{p}$$

$$\Rightarrow (p-1-q+1)d=\frac{p-q}{pq}$$

$$\Rightarrow (p-q)d=\frac{p-q}{pq}$$

$$\Rightarrow d=\frac{1}{pq}$$

Putting the value of d in (1), we obtain

$$a+(p-1)\frac{1}{pq}=\frac{1}{q}$$

$$\Rightarrow a=\frac{1}{q}-\frac{1}{q}+\frac{1}{pq}=\frac{1}{pq}$$

$$\therefore S_{pq}=\frac{pq}{2}\left[2a+(pq-1)d\right]$$

$$=\frac{pq}{2}\left[\frac{2}{pq}+(pq-1)\frac{1}{pq}\right]$$

$$=1+\frac{1}{2}(pq-1)$$

$$=\frac{1}{2}pq+1-\frac{1}{2}=\frac{1}{2}pq+\frac{1}{2}$$

$$=\frac{1}{2}(pq+1)$$

Thus, the sum of first pq terms of the A.P. is $=\frac{1}{2}(pq+1)$.

Question 6:

If the sum of a certain number of terms of the A.P. 25, 22, 19, …… is 116. Find the last term

Solution 6:

Let the sum of n terms of the given A.P. be 116.

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$

Here, $a=25$ and $d=22-25=-3$

$$\therefore S_n=\frac{n}{2}\left[2\times 25+(n-1)(-3)\right]$$

$$\Rightarrow 116=\frac{n}{2}[50-3n+3]$$

$$\Rightarrow 232=n(53-3n)=53n-3n^2$$

$\Rightarrow 3n^2 - 53n + 232 = 0$
$\Rightarrow 3n^2 - 24n - 29n + 232 = 0$
$\Rightarrow 3n(n-8) - 29(n-8) = 0$
$\Rightarrow (n-8)(3n-29) = 0$
$\Rightarrow n = 8 \text{ or } n = \frac{29}{3}$

However, n cannot be equal to $\frac{29}{3}$ therefore, $n = 8$

$\therefore a_8 = \text{Last term} = a + (n-1)d = 25 + (8-1)(-3)$
$= 25 + (7)(-3) = 25 - 21$
$= 4$

Thus, the last term of the A.P. is 4.

Question 7:

Find the sum to n terms of the A.P., whose k^{th} term is $5k+1$.

Solution 7:

It is given that the k^{th} term of the A.P. is $5k+1$.

k^{th} term $= a_k + (k-1)d$
$\therefore a + (k-1)d = 5k+1$
$a + kd - d = 5k + 1$
$\therefore$ Comparing the coefficient of k, we obtain $d = 5$;
$\Rightarrow a - d = 1$
$\Rightarrow a - 5 = 1$
$\Rightarrow a = 6$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$
$= \frac{n}{2}\left[2(6) + (n-1)(5)\right]$
$= \frac{n}{2}[12 + 5n - 5]$
$= \frac{n}{2}[5n + 7]$

Question 8:

If the sum of n terms of an A.P. is $(pn + qn^2)$, where p and q are constants, find the common difference.

Solution 8:

It is known that: $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

According to the given condition,

$$\frac{n}{2}\left[2a+(n-1)d\right] = pn+qn^2$$

$$\Rightarrow \frac{n}{2}\left[2a+nd-d\right] = pn+qn^2$$

$$\Rightarrow na+n^2\frac{d}{2}-n\cdot\frac{d}{2} = pn+qn^2$$

Comparing the coefficients of n^2 on both sides, we obtain

$$\frac{d}{2}=q$$

$$\therefore d=2q$$

Thus, the common difference of the A.P. is $2q$.

Question 9:

The sums of n terms of two arithmetic progressions are in the ratio $5n+4:9n+6$. Find the ratio of their 18^{th} terms.

Solution 9:

Let a_1, a_2 and d_1, d_2 be the first terms and the common difference of the first and second arithmetic progression respectively.

According to the given condition,

$$\frac{\text{Sum of } n \text{ terms of first A.P.}}{\text{Sum of } n \text{ terms of second A.P.}} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \frac{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}{\frac{n}{2}\left[2a_2+(n-1)d_2\right]} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{5n+4}{9n+5} \quad ....(1)$$

Substituting $n=35$ in (1), we obtain

$$\frac{2a_1+34d_1}{2a_2+34d_2} = \frac{5(35)+4}{9(35)+6}$$

$$\Rightarrow \frac{a_1+17d_1}{a_2+17d_2} = \frac{179}{321} \quad ....(2)$$

$$\frac{18^{th} \text{ term of first}}{18^{th} \text{ term of second A.P.}} = \frac{a_1+17d_1}{a_2+17d_2} \quad ....(3)$$

From (2) and (3), we obtain

$$\frac{18^{th} \text{ term of first}}{18^{th} \text{ term of second A.P.}} = \frac{179}{321}$$

Thus, the ratio of 18^{th} term of both the A.P.s is 179 : 321.

Question 10:

If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first $(p+q)$ terms.

Solution 10:

Let a and d be the first term and the common difference of the A.P. respectively.

Here,

$$S_P = \frac{p}{2}\left[2a+(p-1)d\right]$$

$$S_q = \frac{p}{2}\left[2a+(q-1)d\right]$$

According to the given condition,

$$\frac{p}{2}\left[2a+(p-1)d\right] = \frac{q}{2}\left[2a+(q-1)d\right]$$

$$\Rightarrow p\left[2a+(p-1)d\right] = q\left[2a+(q-1)d\right]$$

$$\Rightarrow 2ap + pd(p-1) = 2aq + qd(q-1)$$

$$\Rightarrow 2a(p-q) + d\left[p(p-1) - q(q-1)\right] = 0$$

$$\Rightarrow 2a(p-q) + d\left[p^2 - p - q^2 + q\right] = 0$$

$$\Rightarrow 2a(p-q) + d\left[(p-q)(p+q) - (p-q)\right] = 0$$

$$\Rightarrow 2a(p-q) + d\left[(p-q)(p+q-1)\right] = 0$$

$$\Rightarrow 2a + d(p+q-1) = 0$$

$$\Rightarrow d = \frac{-2a}{p+q-1} \qquad ......(1)$$

$$\therefore\ S_{p+q} = \frac{p+q}{2}\left[2a+(p+q-1)\cdot d\right]$$

$$\Rightarrow S_{p+q} = \frac{p+q}{2}\left[2a+(p+q-1)\left(\frac{-2a}{p+q-1}\right)\right] \qquad \text{[From (1)]}$$

$$= \frac{p+q}{2}\left[2a-2a\right]$$

$$= 0$$

Thus, the sum of the first $(p+q)$ terms of the A.P is 0.

Question 11:

Sum of the first p, q and r terms of an A.P. are a, b and c, respectively. Prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$

Solution 11:

Let a_1 and d be the first term and the common difference of the A.P. respectively.

According to the given information,

$$S_p=\frac{p}{2}\left[2a_1+(p-1)d\right]=a$$

$$\Rightarrow 2a_1+(p-1)d=\frac{2a}{p} \qquad ....(1)$$

$$S_q=\frac{q}{2}\left[2a_1+(q-1)d\right]=b$$

$$\Rightarrow 2a_1+(q-1)d=\frac{2b}{q} \qquad ......(2)$$

$$S_r=\frac{r}{2}\left[2a_1+(r-1)d\right]=c$$

$$\Rightarrow 2a_1+(r-1)d=\frac{2c}{r} \qquad ......(3)$$

Subtracting (2) from (1), we obtain

$$(p-1)d-(q-1)d=\frac{2a}{p}-\frac{2b}{q}$$

$$\Rightarrow d(p-1-q+1)=\frac{2aq-2bp}{pq}$$

$$\Rightarrow d(p-q)=\frac{2aq-2bp}{pq}$$

$$\Rightarrow d=\frac{2(aq-bp)}{pq(p-q)} \qquad ......(4)$$

Subtracting (3) from (2), we obtain

$$(q-1)d-(r-1)d=\frac{2b}{q}-\frac{2c}{r}$$

$$\Rightarrow d(q-1-r+1)=\frac{2b}{q}-\frac{2c}{r}$$

$$\Rightarrow d(q-r)=\frac{2br-2qc}{qr}$$

$$\Rightarrow d=\frac{2(br-qc)}{qr(q-r)} \qquad ......(5)$$

Equating both the values of d obtained in (4) and (5), we obtain

$$\frac{aq-bp}{pq(p-q)}=\frac{br-qc}{qr(q-r)}$$

$\Rightarrow qr(q-r)(aq-bq)=pq(q-q)(br-qc)$

$\Rightarrow r(aq-bp)(q-r)=p(br-qc)(p-q)$

$\Rightarrow (aqr-bpr)(q-r)=(bpr-pqc)(p-q)$

Dividing both sides by pqr, we obtain

$$\left(\frac{a}{p}-\frac{b}{q}\right)(q-r)=\left(\frac{b}{q}-\frac{c}{r}\right)(p-q)$$

$$\Rightarrow \frac{a}{p}(q-r)-\frac{b}{q}(q-r+p-q)+\frac{c}{r}(p-q)=0$$

$$\Rightarrow \frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

Thus, the given result is proved.

Question 12:

The ratio of the sums of m and n terms of an A.P. is $m^2:n^2$. Show that the ratio of m^{th} and n^{th} term is $(2m-1):(2n-1)$.

Solution 12:

Let a and b be the first term and the common difference of the A.P. respectively. According to the given condition,

$$\frac{\text{Sum of } m \text{ terms}}{\text{Sum of } n \text{ terms}}=\frac{m^2}{n^2}$$

$$\Rightarrow \frac{\frac{m}{2}\left[2a+(m-1)d\right]}{\frac{n}{2}\left[2a+(n-1)d\right]}=\frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d}=\frac{m}{n} \qquad ....(1)$$

Putting $m=2m-1$ and $n=2n-1$, we obtain

$$\frac{2a+(2m-2)d}{2a+(2n-2)d}=\frac{2m-1}{2n-1}$$

$$\Rightarrow \frac{a+(m-1)d}{a+(n-1)d}=\frac{2m-1}{2n-1} \qquad .... (2)$$

$$\frac{m^{th} \text{ term of A.P.}}{n^{th} \text{ term of A.P.}}=\frac{a+(m-1)d}{a+(n-1)d} \qquad ......(3)$$

From (2) and (3), we obtain

$$\frac{m^{th} \text{ term of A.P.}}{n^{th} \text{ term of A.P.}}=\frac{2m-1}{2n-1}$$

Thus, the given result is proved.

Question 13:

If the sum of n terms of an A.P. of $3n^2+5n$ and its m^{th} term is 164, find the value of m.

Solution 13:

Let a and b be the first term and the common difference of the A.P. respectively.

$a_m = a+(m-1)d = 164 \qquad ......(1)$

Sum of n terms: $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

Here,

$\frac{n}{2}\left[2a+nd-d\right] = 3n^2+5n$

$\Rightarrow na+n^2\cdot\frac{d}{2}-\frac{nd}{2} = 3n^2+5n$

Comparing the coefficient of n^2 on both sides, we obtain

$\frac{d}{2}=3$

$\Rightarrow d=6$

Comparing the coefficient of n on both sides, we obtain

$a-\frac{d}{2}=5$

$\Rightarrow a-3=5$

$\Rightarrow a=8$

Therefore, from (1), we obtain

$8+(m-1)6=164$

$\Rightarrow (m-1)6=164-8=156$

$\Rightarrow m-1=26$

$\Rightarrow m=27$

Thus, the value of m is 27.

Question 14:

Insert five numbers between 8 and 26 such that resulting sequence is an A.P.

Solution 14:

Let A_1, A_2, A_3, A_4 and A_5 be five numbers between 8 and 26 such that $8, A_1, A_2, A_3, A_4, A_5, 26$ is an A.P.

Here, $a=8,\ b=26, n=7$

Therefore, $26=8+(7-1)d$

$\Rightarrow 6d = 26-8=18$

$\Rightarrow d=3$

$A_1 = a+d = 8+3=11$

$A_2 = a+2d = 8+2\times3 = 8+6 = 14$

$A_3 = a + 3d = 8 + 3\times 3 = 8 + 9 = 17$

$A_4 = a + 4d = 8 + 4\times 3 = 8 + 12 = 20$

$A_5 = a + 5d = 8 + 5\times 3 = 8 + 15 = 23$

Thus, the required five numbers between 8 and 26 are 11, 14, 17, 20 and 23.

Question 15:

If $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b, then find the value of n.

Solution 15:

A.M. of a and b $= \dfrac{a+b}{2}$

According to the given condition,

$\dfrac{a+b}{2} = \dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$

$\Rightarrow (a+b)(a^{n-1} + b^{n-1}) = 2(a^n + b^n)$

$\Rightarrow a^n + ab^{n-1} + ba^{n-1} + b^n = 2a^n + 2b^n$

$\Rightarrow ab^{n-1} + a^{n-1}b = a^n + b^n$

$\Rightarrow ab^{n-1} - b^n = a^n - a^{n-1}b$

$\Rightarrow b^{n-1}(a-b) = a^{n-1}(a-b)$

$\Rightarrow b^{n-1} = a^{n-1}$

$\Rightarrow \left(\dfrac{a}{b}\right)^{n-1} = 1 = \left(\dfrac{a}{b}\right)^0$

$\Rightarrow n - 1 = 0$

$\Rightarrow n = 1$

Question 16:

Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7th and $(m-1)^{th}$ numbers is 5:9. Find the value of m.

Solution 16:

Let $A_1, A_2, ...A_m$ be m numbers such that $1, A_1, A_2, ...A_m, 31$ is an A.P.

Here, $a = 1,\ b = 31,\ n = m + 2$

$\therefore 31 = 1 + (m + 2 - 1)(d)$

$\Rightarrow 30 = (m+1)d$

$\Rightarrow d = \dfrac{30}{m+1}$ ......(1)

$A_1 = a + d$

$A_2 = a + 2d$

$A_3 = a + 3d$

$\therefore\ A_7 = a + 7d$

$A_{m-1} = a + (m-1)d$

According to the given condition,

$$\frac{a+7d}{a+(m-1)d} = \frac{5}{9}$$

$$\Rightarrow \frac{1+7\left(\frac{30}{(m+1)}\right)}{1+(m-1)\left(\frac{30}{m+1}\right)} = \frac{5}{9} \qquad [\text{From}(1)]$$

$$\Rightarrow \frac{m+1+7(30)}{m+1+30(m-1)} = \frac{5}{9}$$

$$\Rightarrow \frac{m+1+210}{m+1+30m-30} = \frac{5}{9}$$

$$\Rightarrow \frac{m+211}{31m-29} = \frac{5}{9}$$

$\Rightarrow 9m + 1899 = 155m - 145$

$\Rightarrow 155m - 9m = 1899 + 145$

$\Rightarrow 146m = 2044$

$\Rightarrow m = 14$

Thus, the value of m is 14.

Question 17:

A man starts repaying a loan as first installment of Rs. 100. If he increases the installment by Rs. 5 every month, what amount he will pay in the 30th installment?

Solution 17:

The first installment of the load is Rs. 100.

The second installment of the load is Rs. 105 and so on.

The amount that the man repays every month forms an A.P.

The A.P. is 100, 105, 110 …

First term, $a = 100$

Common difference, $d = 5$

$A_{30} = a + (30-1)d$

$= 100 + (29)(5)$

$= 100 + 145$

$= 245$

Thus, the amount to be paid in the 30th installment is Rs. 245.

Question 18:

The difference between any two consecutive interior angles of a polygon is $5°$. If the smallest angle is $120°$, find the number of the sides of the polygon.

Solution 18:

The angles of the polygon will form an A.P. with common difference d as $5°$ and first term a as $120°$.

It is known that the sum of all angles of a polygon with n sides is $180(n-2)$.

$\therefore S_n = 180°(n-2)$

$\Rightarrow \frac{n}{2}\left[2a+(n-1)d\right]=180°(n-2)$

$\Rightarrow \frac{n}{2}\left[240°+(n-1)5°\right]=180°(n-2)$

$\Rightarrow n\left[240+(n-1)5\right]=360(n-2)$

$\Rightarrow 240n+5n^2-5n=360n-720$

$\Rightarrow 5n^2-125n+720=0$

$\Rightarrow n^2-25n+144=0$

$\Rightarrow n^2-16n-9n+144=0$

$\Rightarrow n(n-16)-9(n-16)=0$

$\Rightarrow (n-9)(n-16)=0$

$\Rightarrow n=9$ or 16

Vashu Panwar

Exercise 9.3

Question 1:

Find the 20th and n^{th} terms of the G.P. $\frac{5}{2},\frac{5}{4},\frac{5}{8},....$

Solution 1:

The given G.P. is $\frac{5}{2},\frac{5}{4},\frac{5}{8},....$

Here, $a =$ First term $=\frac{5}{2}$

$r =$ Common ratio $=\frac{5/4}{5/2}=\frac{1}{2}$

$a_{20}=ar^{20-1}=\frac{5}{2}\left(\frac{1}{2}\right)^{19}=\frac{5}{(2)(2)^{19}}=\frac{5}{(2)^{20}}$

$a_n=ar^{n-1}=\frac{5}{2}\left(\frac{1}{2}\right)^{n-1}=\frac{5}{(2)(2)^{n-1}}=\frac{5}{(2)^n}$

Question 2:

Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

Solution 2:

Common ratio, r = 2

Let a be the first term of the G.P.

$$\therefore a_8 = ar^{8-1} = ar^7 \Rightarrow ar^7 = 192 \Rightarrow a(2)^7 = 192 \Rightarrow a(7)^7 = (2)^6(3)$$

$$\Rightarrow a = \frac{(2)^6 \times 3}{(2)^7} = \frac{3}{2}$$

$$\therefore a_{12} = ar^{12-1} = \left(\frac{3}{2}\right)(2)^{11} = (3)(2)^{10} = 3072.$$

Question 3:

The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.

Solution 3:

Let a be the first term and r be the common ratio of the G.P. According to the given condition,

$$a_5 = ar^{5-1} = ar^4 = p \text{ ....}(1)$$

$$a_8 = ar^{8-1} = ar^7 = q \text{ ....}(2)$$

$$a_{11} = ar^{11-1} = ar^{10} = s \text{ ....}(3)$$

Dividing equation (2) by (1), we obtain

$$\frac{ar^7}{ar^4} = \frac{q}{p}$$

$$r^3 = \frac{q}{p} \qquad \text{.......}(4)$$

Dividing equation (3) by (2), we obtain

$$\frac{ar^{10}}{ar^7} = \frac{s}{q}$$

$$\Rightarrow r^3 = \frac{s}{q} \qquad \text{......}(5)$$

Equating the values of r^3 obtained in (4) and (5), we obtain

$$\frac{q}{p} = \frac{s}{q}$$

$$\Rightarrow q^2 = ps$$

Thus, the given result is proved.

Question 4:
The 4th term of a G.P. is square of its second term, and the first term is -3. Determine its 7th term.

Solution 4:
Let a be the first term and r be the common ratio of the G.P.
$\therefore a=-3$
It is known that, $a_n=ar^{n-1}$
$\therefore a_4=ar^3=(-3)r^3$
$a_2=a\,r^1=(-3)r$
According to the given condition,
$(-3)r^3=[(-3)r]^2$
$\Rightarrow -3r^3=9r^2 \Rightarrow r=-3 a_7=a\,r^{7-1}=ar^6=(-3)(-3)^6=-(3)^7=-2187$
Thus, the seventh term of the G.P. is -2187.

Question 5:
Which term of the following sequences:
(a) $2,2\sqrt{2},4.....$ is 128?
(b) $\sqrt{3},3,3\sqrt{3},......$ is 729?
(c) $\frac{1}{3},\frac{1}{9},\frac{1}{27},....$ is $\frac{1}{19683}$?

Solution 5:
(a) The given sequence is $2,2\sqrt{2},4.....$ is 128?
Here, $a=2$ and $r=(2\sqrt{2})/2=\sqrt{2}$
Let the n^{th} term of the given sequence be 128.
$a_n=a\,r^{n-1}$
$\Rightarrow (2)(\sqrt{2})^{n-1}=128$
$\Rightarrow (2)(2)^{\frac{n-1}{2}}=(2)^7$
$\Rightarrow (2)^{\frac{n-1}{2}+1}=(2)^7$
$\therefore \frac{n-1}{2}+1=7$
$\Rightarrow \frac{n-1}{2}=6$
$\Rightarrow n-1=12$
$\Rightarrow n=13$
Thus, the 13th term of the given sequence is 128.

(b) The given sequence is $\sqrt{3}, 3, 3\sqrt{3}, ...$

$a = \sqrt{3}$ and $r = \frac{3}{\sqrt{3}} = \sqrt{3}$

Let the n^{th} term of the given sequence be 729.

$a_n = a r^{n-1}$

$\therefore a r^{n-1} = 729$

$\Rightarrow \left(\sqrt{3}\right)\left(\sqrt{3}\right)^{n-1} = 729$

$\Rightarrow (3)^{1/2} (3)^{\frac{n-1}{2}} = (3)^6$

$\Rightarrow (3)^{\frac{1}{2}+\frac{n-1}{2}} = (3)^6$

$\therefore \frac{1}{2} + \frac{n-1}{2} = 6$

$\Rightarrow \frac{1+n-1}{2} = 6$

$\Rightarrow n = 12$

Thus, the 12$^{\text{th}}$ term of the given sequence is 729.

(c) The given sequence is $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} ...$

Here, $a = \frac{1}{3}$ and $r = \frac{1}{9} \div \frac{1}{3} = \frac{1}{3}$

Let the n^{th} term of the given sequence be $\frac{1}{19683}$.

$a_n = a r^{n-1}$

$\therefore a r^{n-1} = \frac{1}{19683}$

$\Rightarrow \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683}$

$\Rightarrow \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^9$

$\Rightarrow n = 9$

Thus, the 9$^{\text{th}}$ term of the given sequence is $\frac{1}{19683}$.

Question 6:

For what values of x, the numbers $\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Solution 6:

The given numbers are $\frac{-2}{7}, x, \frac{-7}{2}$

Common ratio $= \frac{x}{-2/7} = \frac{-7x}{2}$

Also, common ratio $= \frac{-7/2}{x} = \frac{-7}{2x}$

$$\therefore \frac{-7x}{2} = \frac{-7}{2x}$$

$$\Rightarrow x^2 = \frac{-2\times 7}{-2\times 7} = 1$$

$$\Rightarrow x = \sqrt{1}$$

$$\Rightarrow x = \pm 1$$

Thus, for $x = \pm 1$, the given numbers will be in G.P.

Question 7:

Find the sum up to 20 terms in the geometric progression 0.15, 0.015, 0.0015….

Solution 7:

The given G.P. is 0.15, 0.015, 0.00015 …

Here, $a = 0.15$ and $r = \frac{0.015}{0.15} = 0.1$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\therefore S_{20} = \frac{0.15\left[1-(0.1)^{20}\right]}{1-0.1}$$

$$= \frac{0.15}{0.9}\left[1-(0.1)^{20}\right]$$

$$= \frac{15}{90}\left[1-(0.1)^{20}\right]$$

$$= \frac{1}{6}\left[1-(0.1)^{20}\right]$$

Question 8:

Find the sum of n terms in the geometric progression $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Solution 8:

The given G.P. is $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Here, $a = \sqrt{7}$ and $r = \frac{\sqrt{21}}{7} = \sqrt{3}$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\Rightarrow S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}}$$

$$\Rightarrow S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$\Rightarrow S_n = \frac{\sqrt{7}\left(\sqrt{3}+1\right)\left[1-\left(\sqrt{3}\right)^n\right]}{1-3}$$

$$\Rightarrow S_n = \frac{-\sqrt{7}\left(\sqrt{3}+1\right)\left[1-\left(\sqrt{3}\right)^n\right]}{2}$$

$$\Rightarrow \frac{\sqrt{7}\left(1+\sqrt{3}\right)}{2}\left[(3)^{\frac{n}{2}}-1\right]$$

Question 9:

Find the sum of n terms in the geometric progression $1,-a,a^2,-a^3 \ldots\ldots (\text{if } a \neq -1)$

Solution 9:

The given G.P. is $1,-a,a^2,-a^3 \ldots\ldots$

Here, first term $= a_1 = 1$

Common ratio $= r = -a$

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

$$\therefore S_n = \frac{1\left[1-(-a)^n\right]}{1-(-a)} = \frac{\left[1-(-a)^n\right]}{1+a}$$

Question 10:

Find the sum of n terms in the geometric progression $x^3, x^5, x^7 \ldots\ldots (\text{if } x \neq \pm 1)$

Solution 10:

The given G.P. is $x^3, x^5, x^7 \ldots\ldots$

Here, $a = x^3$ and $r = x^2$

$$S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{x^3\left[1-\left(x^2\right)^n\right]}{1-x^2} = \frac{x^3\left(1-x^{2n}\right)}{1-x^2}$$

Question 11:

Evaluate $\sum_{k=1}^{11}\left(2+3^{k}\right)$

Solution 11:

$\sum_{k=1}^{11}\left(2+3^{k}\right)=\sum_{k=1}^{11}(2)+\sum_{k=1}^{11}\left(3^{k}\right)=22+\sum_{k=1}^{11}3^{k}$ ....(1)

$\sum_{k=1}^{11}3^{k}=3^{1}+3^{2}+3^{3}+\ldots\ldots+3^{11}$

The terms of this sequence $3,3^{2},3^{3}\ldots\ldots$ forms a G.P.

$S_{n}=\dfrac{a\left(r^{n}-1\right)}{r-1}$

$\Rightarrow S_{n}=\dfrac{3\left[(3)^{11}-1\right]}{3-1}$

$\Rightarrow S_{n}=\dfrac{3}{2}\left(3^{11}-1\right)$

$\therefore \sum_{k=1}^{11}3^{k}=\dfrac{3}{2}\left(3^{11}-1\right)$

Substituting this value in equation (1), we obtain

$\sum_{k=1}^{11}\left(2+3^{k}\right)=22+\dfrac{3}{2}\left(3^{11}-1\right)$

Question 12:

The sum of first three terms of a G.P. is $\dfrac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Solution 12:

Let $\dfrac{a}{r}, a, ar$ be the first three terms of the G.P.

$\dfrac{a}{r}+a+ar=\dfrac{39}{10}$ ....(1)

$\left(\dfrac{a}{r}\right)(a)(ar)=1$ ....(2)

From (2), we

Obtain $a^{3}=1$

$\Rightarrow a=1$ (Considering real roots only)

Substituting $a=1$ in equation (1), we obtain

$$\frac{1}{r}+1+r=\frac{39}{10}$$
$$\Rightarrow 1+r+r^2=\frac{39}{10}r$$
$$\Rightarrow 10+10r+10r^2-39r=0$$
$$\Rightarrow 10r^2-29r+10=0$$
$$\Rightarrow 10r^2-25r-4r+10=0$$
$$\Rightarrow 5r(2r-5)-2(2r-5)=0$$
$$\Rightarrow (5r-2)(2r-5)=0$$
$$\Rightarrow r=\frac{2}{5} \text{ or } \frac{5}{2}$$

Thus, the three terms of G.P. are $\frac{5}{2}, 1$ and $\frac{2}{5}$.

Question 13:

How many terms of G.P. $3, 3^2, 3^3 \ldots$ are needed to give the sum 120?

Solution 13:

The given G.P. is $3, 3^2, 3^3 \ldots$

Let n terms of this G.P. be required to obtain in the sum as 120.

$$S_n=\frac{a(1-r^n)}{1-r}$$

Here, $a=3$ and $r=3$

$$\therefore S_n=120=\frac{3(3^n-1)}{3-1}$$
$$\Rightarrow 120=\frac{3(3^n-1)}{2}$$
$$\Rightarrow \frac{120\times 2}{3}=3^n-1$$
$$\Rightarrow 3^n-1=80$$
$$\Rightarrow 3^n=81$$
$$\Rightarrow 3^n=3^4$$
$$\therefore n=4$$

Thus, four terms of the given G.P. are required to obtain the sum as 120.

Question 14:

The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

Solution 14:

Let the G.P. be $a, ar, ar^2, ar^3, ...$ According to the given condition,

$a + ar + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$\Rightarrow a(1 + r + r^2) = 16 \qquad ......(1)$

$ar^3(1 + r + r^2) = 128 \qquad ......(2)$

Dividing equation (2) by (1), we obtain

$$\frac{ar^3(1+r+r^3)}{a(1+r+r^2)} = \frac{128}{16}$$

$\Rightarrow r^3 = 8$

$\therefore r = 2$

Substituting $r = 2$ in (1), we obtain $a(1+2+4) = 16$

$\Rightarrow a(7) = 16$

$\Rightarrow a = \frac{16}{7}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_n = \frac{16}{7}\frac{(2^n - 1)}{2 - 1} = \frac{16}{7}(2^n - 1)$$

Question 15:

Given a G.P. with $a = 729$ and 7^{th} term 64, determine S_7.

Solution 15:

$a = 729 \;\; a_7 = 64$

Let r be the common ratio of the G.P. It is known that,

$a_n = a r^{n-1}$

$a_7 = ar^{7-1} = (729) r^6$

$\Rightarrow 64 = 729 r^6$

$\Rightarrow r^6 = \left(\frac{2}{3}\right)^6$

$\Rightarrow r = \frac{2}{3}$

Also, it is known that,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore S_7 = \frac{729\left(1-\left(\frac{2}{3}\right)^7\right)}{1-\frac{2}{3}}$$
$$= 3\times 729\left[1-\left(\frac{2}{3}\right)^7\right]$$
$$= (3)^7\left[\frac{(3)^7-(2)^7}{(3)^7}\right]$$
$$= (3)^7-(2)^7$$
$$= 2187-128$$
$$= 2059$$

Question 16:

Find a G.P. for which sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution 16:

Let a be the first term and r be the common ratio of the G.P.

According to the given conditions,

$$A_2 = -4 = \frac{a\left(1-r^2\right)}{1-r} \qquad \text{......}(1)$$
$$a_5 = 4\times a_3$$
$$\Rightarrow ar^4 = 4ar^2 \Rightarrow r^2 = 4$$
$$\therefore r = \pm 2$$

From (1), we obtain

$$-4 = \frac{a\left[1-(2)^2\right]}{1-2} \text{ for } r = 2$$
$$\Rightarrow -4 = \frac{a(1-4)}{-1}$$
$$\Rightarrow -4 = a(3)$$
$$\Rightarrow a = \frac{-4}{3}$$

Also, $-4 = \dfrac{a\left[1-(-2)^2\right]}{1-(-2)}$ for $r = -2$

$$\Rightarrow -4 = \frac{a(1-4)}{1+2}$$

$\Rightarrow -4 = \frac{a(-3)}{3}$

$\Rightarrow a = 4$

Thus, the required G.P. is $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$ or $4, -8, -16, -32\ldots$

Question 17:

If the 4th, 10th and 16th terms of a G.P. are x, y, and z, respectively. Prove that x, y, z are in G.P.

Solution 17:

Let a be the first term and r be the common ratio of the G.P.

According to the given condition,

$a_4 = ar^3 = x \quad \ldots\ldots(1)$

$a_{10} = ar^9 = y \quad \ldots\ldots(2)$

$a_{16} = ar^{15} = z \quad \ldots\ldots(3)$

Dividing (2) by (1), we obtain

$\frac{y}{x} = \frac{ar^9}{ar^3} \Rightarrow \frac{y}{x} = r^6$

Dividing (3) by (2), we obtain

$\frac{z}{y} = \frac{ar^{15}}{ar^9} \Rightarrow \frac{z}{y} = r^6$

$\therefore \frac{y}{x} = \frac{z}{y}$

Thus, x, y, z are in G.P.

Question 18:

Find the sum to n terms of the sequence, 8, 88, 888, 8888 ….

Solution 18:

The given sequence is 8, 88, 888, 8888 ….

This sequence is not a G.P. However, it can be changed to G.P. by writing the terms as

$S_n = 8 + 88 + 888 + 8888 + \ldots\ldots$ to n terms

$= \frac{8}{9}[9 + 99 + 999 + 9999 + \ldots\ldots \text{to } n \text{ terms}]$

$= \frac{8}{9}\left[(10-1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \ldots\ldots \text{to } n \text{ terms}\right]$

$= \frac{8}{9}\left[(10 + 10^2 + \ldots\ldots\ n \text{ terms}) - (1 + 1 + 1 + \ldots\ldots n \text{ terms})\right]$

$$=\frac{8}{9}\left[\frac{10\left(10^{n}-1\right)}{10-1}-n\right]$$

$$=\frac{8}{9}\left[\frac{10\left(10^{n}-1\right)}{9}-n\right]$$

$$=\frac{80}{81}\left(10^{n}-1\right)-\frac{8}{9}n$$

Question 19:

Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, 1/2.

Solution 19:

Required sum $=2\times128+4\times32+8\times8+16\times2+32\times\frac{1}{2}$

$$=64\left[4+2+1+\frac{1}{2}+\frac{1}{2^{2}}\right]$$

Here, $4,2,1,\frac{1}{2},\frac{1}{2^{2}}$ is a G.P.

First term, $a=4$

Common ratio, $r=\frac{1}{2}$

It is known that, $S_{n}=\frac{a\left(1-r^{n}\right)}{1-r}$

$$\therefore S_{5}=\frac{4\left[1-\left(\frac{1}{2}\right)^{5}\right]}{1-\frac{1}{2}}=\frac{4\left[1-\frac{1}{32}\right]}{\frac{1}{2}}=8\left(\frac{32-1}{32}\right)=\frac{31}{4}$$

$\therefore$ Required sum $=64\left(\frac{31}{4}\right)=(16)(31)=496$

Question 20:

Show that the products of the corresponding terms of the sequences form $a,ar,ar^{2},.....ar^{n-1}$ and $A,AR,AR^{2},\ AR^{n-1}$ a G.P., and find the common ratio.

Solution 20:

It has to be proved that the sequence: $aA,arAR,ar^{2}AR^{2},\ ....ar^{n-1}\ AR^{n-1}$, forms a G.P.

$$\frac{\text{Second term}}{\text{First term}}=\frac{ar\ AR}{aA}=rR$$

$$\frac{\text{Third term}}{\text{Second term}} = \frac{ar^2\,AR^2}{arAR} = rR$$

Thus, the above sequence forms a G.P. and the common ratio is rR.

Question 21:

Find four numbers forming a geometric progression in which third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

Solution 21:

Let a be the first term and r be the common ratio of the G.P.

$a_1 = a,\ a_2 = ar,\ a_3 = ar^2,\ a_4 = ar^3$

By the given condition,

$a_3 = a_1 + 9 \Rightarrow ar^2 = a + 9 \qquad .....(1)$

$a_4 = a_4 + 18 \Rightarrow ar = ar^3 + 18 \qquad .....(2)$

From (1) and (2), we obtain

$a\left(r^2 - 1\right) = 9.......(3)$

$ar\left(1 - r^2\right) = 18.......(4)$

Dividing (4) by (3), we obtain

$$\frac{ar\left(1-r^2\right)}{a\left(r^2-1\right)} = \frac{18}{9}$$

$\Rightarrow -r = 2$

$\Rightarrow r = -2$

Substituting the value of r in (1), we obtain

$4a = a + 9$

$\Rightarrow 3a = 9$

$\therefore a = 3$

Thus, the first four numbers of the G.P. are $3,\ 3(-2),\ 3(-2)^2$, and $3(-2)^3$

i.e., $3, -6, 12$ and -24.

Question 22:

If p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$.

Solution 22:

Let A be the first term and R be the common ratio of the G.P.

According to the given information,

$AR^{p-1} = a$

$AR^{q-1} = b$

$AR^{r-1} = c$

$a^{q-r}\cdot b^{r-p}\cdot c^{p-q}$
$=A^{q-r}\times R^{(p-1)(q-r)}\times A^{r-p}\times R^{(q-1)(r-p)}\times A^{p-q}\times R^{(r-1)(p-q)}$
$=A^{q-r+r-p+p-q}\times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)}$
$=A^0\times R^0$
$=1$
Thus, the given result is proved.

Question 23:

If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2=(ab)^n$.

Solution 23:

The first term of the G.P is a and the last term is b.
Therefore, the G.P. is $a, ar, ar^2, ar^3 .... ar^{n-1}$, where r is the common ratio.

$b=ar^{n-1}$ ....(1)

$P=$ Product of n terms
$=(a)(ar)(ar^2)......(ar^{n-1})$
$=(a\times a\times ....a)(r\times r^2\times ....r^{n-1})$
$=a^n\, r^{1+2+....(n-1)}$ ....(2)

Here, $1,2,.....(n-1)$ is an A.P.
$\therefore 1+2+.......+(n-1)$
$=\frac{n-1}{2}[2+(n-1-1)\times 1]=\frac{n-1}{2}[2+n-2]=\frac{n(n-1)}{2}$

$P=a^n\, r^{\frac{n(n-1)}{2}}$
$\therefore P^2=a^{2n}\, r^{n(n-1)}$
$=\left[a^2\, r^{(n-1)}\right]^n$
$=\left[a\times ar^{n-1}\right]^n$
$=(ab)^n$ $\quad$ [Using (1)]

Thus, the given result is proved.

Question 24:

Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Solution 24:

Let a be the first term and r be the common ratio of the G.P.

Sum of first n terms $=\dfrac{a\left(1-r^n\right)}{(1-r)}$

Since there are n terms from $(n+1)^{th}$ to $(2n)^{th}$ term,

Sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term

$$S_n=\frac{a_{n+1}\left(1-r^n\right)}{1-r}$$

$$a^{n+1}=ar^{n+1-1}=ar^n$$

Thus, required ratio $=\dfrac{a\left(1-r^n\right)}{(1-r)}\times\dfrac{(1-r)}{ar^n\left(1-r^n\right)}=\dfrac{1}{r^n}$

Thus, the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\dfrac{1}{r^n}$.

Question 25:

If a,b,c and d are in G.P. show that:

$$\left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right)=(ab+bc+cd)^2$$

Solution 25:

If a,b,c and d are in G.P. Therefore,

$bc=ad$ ....(1)

$b^2=ac$ ....(2)

$c^2=bd$ ....(3)

It has to be proved that,

$$\left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right)=(ab+bc+cd)^2$$

R.H.S.

$=(ab+bc+cd)^2$

$=(ab+ad+cd)^2$ $\quad$ [Using (1)]

$=\left[ab+d(a+c)\right]^2$

$=a^2b^2+2abd(a+c)+d^2(a+c)^2$

$=a^2b^2+2a^2bd+2acbd+d^2\left(a^2+2ac+c^2\right)$

$=a^2b^2+2a^2c^2+2b^2c^2+d^2a^2+2d^2b^2+d^2c^2$ $\quad$ [Using (1) and (2)]

$=a^2b^2+a^2c^2+a^2c^2+b^2c^2+b^2c^2+d^2a^2+d^2b^2+d^2b^2+d^2c^2$

$= a^2b^2 + a^2c^2 + a^2d^2 + b^2 \times b^2 + b^2c^2 + b^2d^2 + c^2b^2 + c^2 \times c^2 + c^2d^2$

[Using (2) and (3) and rearranging terms]

$= a^2\left(b^2 + c^2 + d^2\right) + b^2\left(b^2 + c^2 + d^2\right) + c^2\left(b^2 + c^2 + d^2\right)$

$= \left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right) = \text{L.H.S}$

$\therefore$ L.H.S = R.H.S.

$\therefore \left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right) = \left(ab + bc + cd\right)^2$.

Question 26:

Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Solution 26:

Let G_1 and G_2 be two numbers between 3 and 81 such that the series, 3, $G_1, G_2, 81$, forms a G.P.

Let a be the first term and r be the common ratio of the G.P.

$\therefore\ 81 = (3)(r)^3$

$\Rightarrow r^3 = 27$

$\therefore r = 3$ (Talking real roots only)

For $r = 3$,

$G_1 = ar = (3)(3) = 9$

$G_2 = ar^2 = (3)(3)^2 = 27$

Thus, the required two numbers are 9 and 27.

Question 27:

Find the value of n so that $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

Solution 27:

M. of a and b is $\sqrt{ab}$

By the given condition: $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$

Squaring both sides, we obtain

$$\frac{\left(a^{n+1} + b^{n+1}\right)^2}{\left(a^n + b^n\right)^2} = ab$$

$\Rightarrow a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = (ab)\left(a^{2n} + 2a^nb^n + b^{2n}\right)$

$\Rightarrow a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = a^{2n+1}b + 2a^{n+1}b^{n+1} + ab^{2n+1}$

$\Rightarrow a^{2n+2} + b^{2n+2} = a^{2n+1}b + ab^{2n+1}$

$\Rightarrow a^{2n+2} - a^{2n+1}b = ab^{2n+1} - b^{2n+2}$

$$\Rightarrow a^{2n+1}(a-b)=b^{2n+1}(a-b)$$
$$\Rightarrow \left(\frac{a}{b}\right)^{2n+1}=1=\left(\frac{a}{b}\right)^{0}$$
$$\Rightarrow 2n+1=0$$
$$\Rightarrow n=\frac{-1}{2}$$

Question 28:

The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$

Solution 28:

Le the two numbers be a and b.

G.M. $=\sqrt{ab}$

According to the given condition,

$$a+b=6\sqrt{ab} \qquad ......(1)$$
$$\Rightarrow (a+b)^2=36(ab)$$

Also,

$$(a-b)^2=(a+b)^2-4ab=36ab-4ab=32ab$$
$$\Rightarrow a-b=\sqrt{32}\sqrt{ab}$$
$$=4\sqrt{2}\sqrt{ab} \qquad ......(2)$$

Adding (1) and (2), we obtain

$$2a=\left(6+4\sqrt{2}\right)\sqrt{ab}$$
$$a=\left(3+2\sqrt{2}\right)\sqrt{ab}$$

Substituting the value of a in (1), we obtain

$$b=6\sqrt{ab}-\left(3+2\sqrt{2}\right)\sqrt{ab}$$
$$\Rightarrow b=\left(3-2\sqrt{2}\right)\sqrt{ab}$$
$$\frac{a}{b}=\frac{\left(3+2\sqrt{2}\right)\sqrt{ab}}{\left(3-2\sqrt{2}\right)\sqrt{ab}}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

Thus, the required ratio is $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Question 29:

If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$

Solution 29:

It is given that A and G are A.M. and G.M. between two positive numbers.
Let these two positive numbers be a and b.

$\therefore AM = A = \frac{a+b}{2}$ .....(1)

$GM = G = \sqrt{ab}$ ....(2)

From (1) and (2), we obtain

$a+b=2A$ .....(3)

$ab=G^2$ ....(4)

Substituting the value of a and b from (3) and (4) in the identity

$(a-b)^2=(a+b)^2-4ab$,

We obtain

$(a-b)^2=4A^2-4G^2=4(A^2-G^2)$

$(a-b)^2=4(A+G)(A-G)$

$(a-b)=2\sqrt{(A+G)(A-G)}$ ......(5)

From (3) and (5), we obtain

$2a=2A+2\sqrt{(A+G)(A-G)}$

$\Rightarrow a=A+\sqrt{(A+G)(A-G)}$

Substituting the value of a in (3), we obtain

$b=2A-A-\sqrt{(A+G)(A-G)}=A-\sqrt{(A+G)(A-G)}$

Thus, the two numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Question 30:

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and n^{th} hour?

Solution 30:

It is given that the number of bacteria doubles every hour. Therefore, the number of bacteria after every hour will form a G.P.

Here, $a=30$ and $r=2$ $\therefore a_3=ar^2=(30)(2)^2=120$

Therefore, the number of bacteria at the end of 2nd hour will be 120.

$a_5 = ar^4 = (30)(2)^4 = 480$

The number of bacteria at the end of 4th hour will be 480.

$a_{n+1} = ar^n = (30)2^n$

Thus, number of bacteria at the end of n^{th} hour will be $30(2)^n$.

Question 31:

What will Rs. 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Solution 31:

The amount deposited in the bank is Rs. 500.

At the end of first year, amount $= \text{Rs.}500\left(1+\frac{1}{10}\right) = \text{Rs.}500(1.1)$

At the end of 2nd year, amount = Rs. 500 (1.1) (1.1)

At the end of 3rd year, amount = Rs. 500 (1.1) (1.1) (1.1) and so on

$\therefore$ Amount at the end of 10 years = Rs. 500 (1.1) (1.1) ….. (10 times)

$= \text{Rs.}500(1.1)^{10}$.

Question 32:

If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Solution 32:

Let the root of the quadratic equation be a and b.

According to the given condition,

A.M. $= \frac{a+b}{2} = 8 \Rightarrow a+b = 16$ .....(1)

G.M. $= \sqrt{ab} = 5 \Rightarrow ab = 25$ ......(2)

The quadratic equation is given by,

$x^2 - x(\text{Sum of roots}) + (\text{Product of roots}) = 0$

$x^2 - x(a+b) + (ab) = 0$

$x^2 - 16x + 25 = 0$ [Using (1) and (2)]

Thus, the required quadratic equation is $x^2 - 16x + 25 = 0$.

Exercise 9.4

Question 1:

Find the sum to n terms of the series $1\times2+2\times3+3\times4+4\times5+....$

Solution 1:

The given series is $1\times2+2\times3+3\times4+4\times 5+.....n^{th}$ term, $a_n=n(n+1)$

$$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k(k+1)$$

$$=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)$$

$$=\frac{n(n+1)}{2}\left(\frac{2n+4}{3}\right)$$

$$=\frac{n(n+1)(n+2)}{3}$$

Question 2:

Find the sum to n terms of the series $1\times2\times3+2\times3\times4+3\times4\times5+....$

Solution 2:

The given series is $1\times2\times3+2\times3\times4+3\times4\times5+.....n^{th}$ term,

$$a_n=n(n+1)(n+2)$$

$$=(n^2+n)(n+2)$$

$$=n^3+3n^2+2n$$

$$\therefore S_n=\sum_{k=1}^{n}a_k$$

$$=\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$

$$=\left[\frac{n(n+1)}{2}\right]^2+\frac{3n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}$$

$$=\left[\frac{n(n+1)}{2}\right]^2+\frac{n(n+1)(2n+1)}{2}+n(n+1)$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+2n+1+2\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{n^2+n+4n+6}{2}\right]$$

$$=\frac{n(n+1)}{4}\left(n^2+5n+6\right)$$

$$=\frac{n(n+1)}{4}\left(n^2+2n+3n+6\right)$$

$$=\frac{n(n+1)\left[n(n+2)+3(n+2)\right]}{4}$$

$$=\frac{n(n+1)(n+2)(n+3)}{4}$$

Question 3:

Find the sum to n terms of the series $3\times1^2+5\times2^2+7\times3^2+\ldots$

Solution 3:

The given series is $3\times1^2+5\times2^2+7\times3^2+\ldots\ldots n^{th}$ term,

$$a_n=(2n+1)n^2=2n^3+n^2$$

$$\therefore S_n=\sum_{k=1}^{n}a_k$$

$$=\sum_{k=1}^{n}=\left(2k^3+k^2\right)=2\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2$$

$$=2\left[\frac{n(n+1)}{2}\right]^2+\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n^2(n+1)^2}{2}+\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n(n+1)}{2}\left[n(n+1)+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+5n+1}{3}\right]$$

$$=\frac{n(n+1)\left(3n^2+5n+1\right)}{6}$$

Question 4:

Find the sum to n terms of the series $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+.....$

Solution 4:

The given series is $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+.....$

n^{th} term, $a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ [By partial fractions]

$a_1=\frac{1}{1}-\frac{1}{2}$

$a_2=\frac{1}{2}-\frac{1}{3}$

$a_3=\frac{1}{3}-\frac{1}{4}....$

$a_n=\frac{1}{n}-\frac{1}{n+1}$

Adding the above terms column wise, we obtain

$a_1+a_2+....+a_n=\left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....\frac{1}{n}\right]-\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....\frac{1}{n+1}\right]$

$\therefore S_n=1-\frac{1}{n+1}=\frac{n+1-1}{n+1}=\frac{n}{n+1}$

Question 5:

Find the sum to n terms of the series $5^2+6^2+7^2+.....+20^2$

Solution 5:

The given series is $5^2+6^2+7^2+.....+20^2$ n^{th} term,

$a_n=(n+4)^2=n^2+8n+16$

$\therefore\ S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}(k^2+8k+16)$

$=\sum_{k=1}^{n}k^2+8\sum_{k=1}^{n}k+\sum_{k=1}^{n}16$

$=\frac{n(n+1)(2n+1)}{6}+\frac{8n(n+1)}{2}+16n$

16^{th} term is $(16+4)^2=20^2$

$\therefore S_{16}=\frac{16(16+1)(2\times16+1)}{6}+\frac{8\times16\times(16+1)}{2}+16\times16$

$=\frac{(16)(17)(33)}{6}+\frac{(8)\times16\times(16+1)}{2}+16\times16$

$=\frac{(16)(17)(33)}{6}+\frac{(8)(16)(17)}{2}+256$

$=1496+1088+256$

$=2840$

$\therefore 5^2+6^2+7^2+.......+20^2=2840$.

Question 6:

Find the sum to n terms of the series $3\times8+6\times11+9\times14+...$

Solution 6:

The given series is $3\times8+6\times11+9\times14+.....a_n$

$=(n^{th}\text{ term of }3,6,9....)\times(n^{th}\text{ term of }8,11,14....)$

$=(3n)(3n+5)$

$=9n^2+15n$

$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}(9k^2+15k)$

$=\sum_{k=1}^{n}k^2=15\sum_{k=1}^{n}k$

$=9\times\frac{n(n+1)(2n+1)}{6}+15\times\frac{n(n+1)}{2}$

$=\frac{3n(n+1)(2n+1)}{2}+\frac{15n(n+1)}{2}$

$=\frac{3n(n+1)}{2}(2n+1+5)$

$=\frac{3n(n+1)}{2}(2n+6)$

$=3n(n+1)(n+3)$

Question 7:

Find the sum to n terms of series $1^2+(1^2+2^2)+(1^2+2^2+3^2)+....$

Solution 7:

The given series is $1^2+(1^2+2^2)+(1^2+2^2+3^2)+....a_n$

$=(1^2+2^2+3^3+.....+n^2)$

$=\frac{n(n+1)(2n+1)}{6}$

$$= \frac{n(2n^2+3n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\therefore S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n}\left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right)$$

$$= \frac{1}{3}\sum_{k=1}^{n}k^3 + \frac{1}{2}\sum_{k=1}^{n}k^2 + \frac{1}{6}\sum_{k=1}^{n}k$$

$$= \frac{1}{3}\frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2}\times\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\times\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+1+1}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+2}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)+2(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{(n+1)(n+2)}{2}\right]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

Question 8:

Find the sum to n terms of the series whose n^{th} term is given by $n(n+1)(n+4)$.

Solution 8:

$$a_n = n(n+1)(n+4) = n(n^2+5n+4) = n^3+5n^2+4n$$

$$\therefore\ S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}k^3 + 5\sum_{k=1}^{n}k^2 + 4\sum_{k=1}^{n}k$$

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+23n+34}{6}\right]$$

$$=\frac{n(n+1)(3n^2+23n+34)}{12}$$

Question 9:

Find the sum to n terms of these series whose n^{th} terms is given by n^2+2^n

Solution 9:

$a_n=n^2+2^n$

$$\therefore S_n=\sum_{k=1}^{n}k^2+2^k=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}2^k \qquad ......(1)$$

Consider $\sum_{k=1}^{n}2^k=2^1+2^2+2^3+....$

The above series $2^2+2^3....$ is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^{n}2^k=\frac{(2)\left[(2)^n-1\right]}{2-1}=2(2^n-1) \; ....(2)$$

Therefore, from (1) and (2), we obtain

$$S_n=\sum_{k=1}^{n}k^2+2(2^n-1)=\frac{n(n+1)(2n+1)}{6}+2(2^n-1)$$

Question 10:

Find the sum to n terms of the series whose n^{th} terms is given by $(2n-1)^2$

Solution 10:

$a_n=(2n-1)^2=4n^2-4n+1$

$$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}(4k^2-4k+1)$$

$$=4\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n$$

$$=\frac{2n(n+1)(2n+1)}{3}-2n(n+1)+n$$

$$=n\left[\frac{2\left(2n^2+3n+1\right)}{3}-2\left(n+1\right)+1\right]$$

$$=n\left[\frac{4n^2+6n+2-6n-6+3}{3}\right]$$

$$=n\left[\frac{4n^2-1}{3}\right]$$

$$=\frac{n\left(2n+1\right)\left(2n-1\right)}{3}$$

Miscellaneous Exercise

Question 1:

Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Solution 1:

Let a and d be the first term and the common difference of the A.P. respectively. It is known that the k^{th} term of an A.P. is given by

$$a_k=a+\left(k-1\right)d$$

$$\therefore a_{m+n}=a+\left(m+n-1\right)d$$

$$a_{m-n}=a+\left(m-n-1\right)d$$

$$a_m=a+\left(m-1\right)d$$

$$\therefore a_{m+n}+a_{m-n}=a+\left(m+n-1\right)d+a+\left(m-n-1\right)d$$

$$=2a+\left(m+n-1+m-n-1\right)d$$

$$=2a+\left(2m-2\right)d$$

$$=2a+2\left(m-1\right)d$$

$$=2\left[a+\left(m-1\right)d\right]$$

$$=2a_m$$

Thus, the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Question 2:

Let the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Solution 2:

Let the three numbers in A.P. be $a-d, a,$ and $a+d$.

According to the given information,

$(a-d)+(a)+(a+d)=24$ ......(1)

$\Rightarrow 3a=24$

$\therefore a=8$

$(a-d)a(a+d)=440$ ......(2)

$\Rightarrow (8-d)(8)(8+d)=440$

$\Rightarrow (8-d)(8+d)=55$

$\Rightarrow 64-d^2=55$

$\Rightarrow d^2=64-55=9$

$\Rightarrow d^2=\pm 3$

Therefore, when $d=3$, the numbers are 5, 8 and 11 and when $d=-3$, the numbers are 11, 8 and 5.

Thus, the three numbers are 5, 8 and 11.

Question 3:

Let the sum of $n, 2n, 3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3=3(S_2-S_1)$

Solution 3:

Let a and b be the first term and the common difference of the A.P. respectively. Therefore,

$S_1=\frac{n}{2}[2a+(n-1)d]$ .... (1)

$S_2=\frac{2n}{2}[2a+(2n-1)d]=n[2a+(2n-1)d]$ ....(2)

$S_3=\frac{3n}{2}[2a+(3n-1)d]$ ....(3)

From (1) and (2), we obtain

$S_2-S_1=n[2a+(2n-1)d]-\frac{n}{2}[2a+(n-1)d]$

$=n\left\{\frac{4a+4nd-2d-2a-nd+d}{2}\right\}$

$=n\left[\frac{2a+3nd-d}{2}\right]$

$=\frac{n}{2}[2a+(3n-1)d]$

$\therefore 3(S_2-S_1)=\frac{3n}{2}[2a+(3n-1)d]=S_3$ [From (3)]

Hence, the given result is proved.

Question 4:

Find the sum of all numbers between 200 and 400 which are divisible by 7.

Solution 4:

The numbers lying between 200 and 400, which are divisible by 7, are 203, 210, 217…. 399

$\therefore$ First term, $a = 203$

Last term, $I = 399$

Common difference, $d = 7$

Let the number of terms of the A.P. be n.

$\therefore a_n = 399 = a + (n-1)d$

$\Rightarrow 399 = 203 + (n-1)7$

$\Rightarrow 7(n-1) = 196$

$\Rightarrow n-1 = 28$

$\Rightarrow n = 29$

$\therefore S_{29} = \frac{29}{2}(203+399)$

$= \frac{29}{2}(602)$

$= (29)(301)$

$= 8729$

Thus, the required sum is 8729.

Question 5:

Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Solution 5:

The integers from 1 to 100, which are divisible by 2, are 2, 4, 6 ….. 100.

This forms an A.P. with both the first term and common difference equal to 2.

$\Rightarrow 100 = 2 + (n-1)2$

$\Rightarrow n = 50$

$\therefore 2+4+6+......+100 = \frac{50}{2}[2(2)+(50-1)(2)]$

$= \frac{50}{2}[4+98]$

$= (25)(102)$

$= 2550$

The integers from 1 to 100, which are divisible by 5, 10…. 100.

This forms an A.P. with both the first term and common difference equal to 5.

$\therefore 100 = 5 + (n-1)5$

$\Rightarrow 5n = 100$

$\Rightarrow n = 20$

$$\therefore 5+10+....+100=\frac{20}{2}\left[2(5)+(20-1)5\right]$$
$$=10\left[10+(19)5\right]$$
$$=10[10+95]=10\times 105$$
$$=1050$$
The integers, which are divisible by both 2 and 5, are 10, 20, ….. 100.
This also forms an A.P. with both the first term and common difference equal to 10.
$$\therefore\ 100=10+(n-1)(10)$$
$$\Rightarrow 100=10n$$
$$\Rightarrow n=10$$
$$\therefore\ 10+20+....+100=\frac{10}{2}\left[2(10)+(10-1)(10)\right]$$
$$=5[20+90]=5(110)=550$$
$$\therefore\ \text{Required sum } =2550+1050-550=3050$$
Thus, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

Question 6:
Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

Solution 6:
The two-digit numbers, which when divided by 4, yield 1 as remainder, are 13, 17, …97.
This series forms an A.P. with first term 13 and common difference 4.
Let n be the number of terms of the A.P.
It is known that the n^{th} term of an A.P. is given by, $a_n=a+(n-1)d$
$$\therefore 97=13+(n-1)(4)$$
$$\Rightarrow 4(n-1)=84$$
$$\Rightarrow n-1=21$$
$$\Rightarrow n=22$$
Sum of n terms of an A.P. is given by
$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$
$$\therefore\ S_{22}=\frac{22}{2}\left[2(13)+(22-1)(4)\right]$$
$$=11[26+84]$$
$$=1210$$
Thus, the required sum is 1210.

Question 7:

If f is a function satisfying $f(x+y)=f(x).f(y)$ for all $x, y\in N$, such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$ find the value of n.

Solution 7:

It is given that,

$f(x+y)=f(x)\times f(y)$ for all $x, y\in N$ ……. (1)

$f(1)=3$

Taking $x=y=1$ in (1),

We obtain $f(1+1)=f(2)=f(1)f(1)=3\times 3=9$

Similarly,

$f(1+1+1)=f(3)=f(1+2)=f(1)f(2)=3\times 9=27$

$f(4)=f(1+4)=f(1)f(3)=3\times 27=81$

$\therefore$ $f(1), f(2), f(3)$, ……, that is 3, 9, 27,….., forms a G.P. with both the first term and common ratio equal to 3.

It is known that, $S_n=\dfrac{a(r^n-1)}{r-1}$

It is given that, $\sum_{k=1}^{n} f(x)=120$

$\therefore\ 120=\dfrac{3(3^n-1)}{3-1}$

$\Rightarrow 120=\dfrac{3}{2}(3^n-1)$

$\Rightarrow 3^n-1=80$

$\Rightarrow 3^n=81=3^4$

$\therefore n=4$

Thus, the value of n is 4.

Question 8:

The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

Solution 8:

Let the sum of n terms of the G.P. be 315.

It is known that, $S_n=\dfrac{a(r^n-1)}{r-1}$

It is given that the first term a is 5 and common ratio r is 2.

$\therefore 315 = \frac{5(2^n - 1)}{2-1}$

$\Rightarrow 2^n - 1 = 63$

$\Rightarrow 2^n = 64 = (2)^6$

$\Rightarrow n = 6$

$\therefore$ Last term of the G.P. $= 6^{th}$ term $= ar^{6-1} = (5)(2)^5 = (5)(32)$

$= 160$

Thus, the last term of the G.P. is 160.

Question 9:

The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Solution 9:

Let a and r be the first term and the common ratio of the G.P. respectively.

$\therefore a = 1 \qquad a_3 = ar^2 = r^2 \qquad a_5 = ar^4 = r^4$

$\therefore r^2 + r^4 = 90$

$\Rightarrow r^4 + r^2 - 90 = 0$

$\Rightarrow r^2 = \frac{-1+\sqrt{1+360}}{2} = \frac{-1\pm\sqrt{361}}{2} = -10 \text{ or } 9$

$\therefore r = \pm 3$ [Taking real roots]

Thus, the common ratio of the G.P. is ± 3.

Question 10:

The sum of the three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Solution 10:

Let the three numbers in G.P. be $a, ar,$ and ar^2.

From the given condition,

$a + ar + ar^2 = 56$

$\Rightarrow a(1 + r + r^2) = 56 \qquad ......(1)$

$a - 1,\ ar - 7,\ ar^2 - 21$ forms an A.P.

$\therefore (ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$ b

$\Rightarrow ar - a - 6 = ar^2 - ar - 14$

$\Rightarrow ar^2 - 2ar + a = 8$

$\Rightarrow ar^2 - ar - ar + a = 8$

$\Rightarrow a(r^2 + 1 - 2r) = 8$

$\Rightarrow a(r^2 - 1)^2 = 8 \qquad ......(2)$

From (1) and (2), we get

$\Rightarrow 7\left(r^2-2r+1\right)=1+r+r^2$
$\Rightarrow 7r^2-14r+7-1-r-r^2=0$
$\Rightarrow 6r^2-15r+6=0$
$\Rightarrow 6r^2-12r-3r+6=0$
$\Rightarrow 6r\left(r-2\right)-3\left(r-2\right)=0$
$\Rightarrow \left(6r-3\right)\left(r-2\right)=0$
When $r=2,\ a=8$
Therefore, when $r=2$, the three numbers in G.P. are 8, 16 and 32.
When, $r=\frac{1}{2}$, the three numbers in G.P. are 32, 16 and 8.
Thus, in either case, the three required numbers are 8, 16 and 32.

Question 11:
A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Solution 11:
Let the G.P. be $T_1, T_2, T_3, T_4 .... T_{2n}$.
Number of terms $=2n$
According to the given condition,
$T_1+T_2+T_3+....+T_{2n}=5\left[T_1+T_3+....+T_{2n-1}\right]$
$\Rightarrow T_1+T_2+T_3+....+T_{2n}-5\left[T_1+T_3+.....+T_{2n-1}\right]=0$
$\Rightarrow T_2+T_4+....+T_{2n}=4\left[T_1+T_3+.....+T_{2n-1}\right]$
Let the G.P. be $a, ar, ar^2, ar^3 ...$
$\therefore \frac{ar\left(r^n-1\right)}{r-1}=\frac{4\times a\left(r^n-1\right)}{r-1}$
$\Rightarrow ar=4a$
$\Rightarrow r=4$
Thus, the common ratio of the G.P. is 4.

Question 12:
The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Solution 12:
Let the A.P. be $a, a+d, a+2d, a+3d .... a+\left(n-2\right)d, a+\left(n-1\right)d$.
Sum of first four terms $=a+\left(a+d\right)+\left(a+2d\right)+\left(a+3d\right)=4a+6d$
Sum of last four terms
$=\left[a+\left(n-4\right)d\right]+\left[a+\left(n-3\right)d\right]+\left[a+\left(n-2\right)d\right]+\left[a+\left(n-1\right)d\right]$

$=4a+(4n-10)d$

According to the given condition,

$4a+6d=56$

$\Rightarrow 4(11)+6d=56$ [Since $a=11(\text{given})$]

$=6d=12$

$=d=2$

$\therefore 4a+(4n-10)d=112$

$\Rightarrow 4(11)+(4n-10)2=112$

$\Rightarrow (4n-10)2=68$

$\Rightarrow 4n-10=34$

$\Rightarrow 4n=44$

$\Rightarrow n=11$

Thus, the number of terms of the A.P. is 11.

Question 13:

If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$ then show that a,b,c and d are in G.P.

Solution 13:

It is given that,

$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$

$\Rightarrow (a+bx)(b-cx)=(b+cx)(a-bx)$

$\Rightarrow ab-acx+b^2x-bcx^2=ab-b^2x+acx-bcx^2$

$\Rightarrow 2b^2x=2acx$

$\Rightarrow b^2=ac$

$\Rightarrow \frac{b}{a}=\frac{c}{b}$ ......(1)

Also, $\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$

$\Rightarrow (b+cx)(c-dx)=(b-cx)(c+dx)$

$\Rightarrow bc-bdx+c^2x-cdx^2=bc+bdx-c^2x-cdx^2$

$\Rightarrow 2c^2x=2bdx$

$\Rightarrow c^2=bd$

$\Rightarrow \frac{c}{d}=\frac{d}{c}$ ......(2)

From (1) and (2), we obtain

$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$

Thus, a,b,c and d are in G.P.

Question 14:
Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$

Solution 14:
Let the G.P. be $a, ar, ar^2, ar^3 .... ar^{n-1}$
According to the given information,

$$S = \frac{a(r^n - 1)}{r - 1}$$

$$P = a^n \times r^{1+2+...+n-1}$$

$$= a^n r^{\frac{n(n-1)}{2}}$$ [$\because$ Sum of first n natural numbers is $n\frac{(n+1)}{2}$]

$$R = \frac{1}{a} + \frac{1}{ar} + ... + \frac{1}{ar^{n-1}}$$

$$= \frac{r^{n-1} + r^{n-2} + ...r + 1}{ar^{n-1}}$$

$$= \frac{1(r^n - 1)}{(r - 1)} \times \frac{1}{ar^{n-1}}$$ $\left[\because 1, r, ...r^{n-1} \text{ forms a G.P}\right]$

$$= \frac{r^n - 1}{ar^{n-1}(r - 1)}$$

$$\therefore P^2R^n = a^{2n} r^{n(n-)} \frac{(r^n - 1)^n}{a^n r^{n(n-1)} (r - 1)^n}$$

$$= \frac{a^n (r^n - 1)^n}{(r - 1)^n}$$

$$= \left[\frac{a(r^n - 1)}{(r - 1)}\right]^n$$

$$= S^n$$

Hence, $P^2R^n = S^n$

Question 15:
The p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively. Show that $(q - r)a + (r - p)b + (p - q)c = 0$

Solution 15:
Let t and d be the first term and the common difference of the A.P. respectively.
The n^{th} term of an A.P. is given by, $a_n = t + (n - 1)d$

Therefore,

$a_p = t + (p-1)d = a$ ....(1)

$a_q = t + (q-1)d = b$ ....(2)

$a_r = t + (r-1)d = c$ ....(3)

Subtracting equation (2) from (1), we obtain

$(p-1-q+1)d = a-b$

$\Rightarrow (p-q)d = a-b$

$\therefore d = \dfrac{a-b}{p-q}$ ......(4)

Subtracting equation (3) from (2), we obtain

$(q-1-r+1)d = b-c$

$\Rightarrow (q-r)d = b-c$

$\Rightarrow d = \dfrac{b-c}{q-r}$ ......(5)

Equating both the values of d obtained in (4) and (5), we obtain

$\dfrac{a-b}{p-q} = \dfrac{b-c}{q-r}$

$\Rightarrow (a-b)(q-r) = (b-c)(p-q)$

$\Rightarrow aq - bq - ar + br = bp - bq - cp + cq$

$\Rightarrow bp - cp + cq - aq + ar - br = 0$

$\Rightarrow (-aq + ar) + (bp - br) + (-cp + cq) = 0$ (By rearranging terms)

$\Rightarrow -a(q-r) - b(r-p) - c(p-q) = 0$

$\Rightarrow a(q-r) + b(r-p) + c(p-q) = 0$

Thus, the given result is proved.

Question 16:

If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

Solution 16:

It is given that $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$\therefore b\left(\frac{1}{c}+\frac{1}{a}\right) - a\left(\frac{1}{b}+\frac{1}{c}\right) = c\left(\frac{1}{a}+\frac{1}{b}\right) - b\left(\frac{1}{c}+\frac{1}{a}\right)$

$\Rightarrow \dfrac{b(a+c)}{ac} - \dfrac{a(b+c)}{bc} = \dfrac{c(a+b)}{ab} - \dfrac{b(a+c)}{ac}$

$\Rightarrow \dfrac{b^2a+b^2c-a^2b-a^2c}{abc}=\dfrac{c^2a+c^2b-b^2a-b^2c}{abc}$

$\Rightarrow b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$

$\Rightarrow ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$

$\Rightarrow ab(b-a)+c(b-a)(b+a)=a(c-b)(c+b)+bc(c-b)$

$\Rightarrow (b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$

$\Rightarrow b-a=c-b$

Thus, a,b and c are in A.P.

Question 17:

If a,b,c,d are in G.P., prove that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P.

Solution 17:

It is given that a,b,c and d are in G.P.

$\therefore b^2=ac$ ......(1)

$c^2=bd$ ....(2)

$ad=bc$ ......(3)

It has to be proved that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P. i.e.,

$(b^n+c^n)^2=(a^n+b^n),(c^n+d^n)$

Consider L.H.S.

$(b^n+c^n)^2=b^{2n}+2b^nc^n+c^{2n}$

$=(b^2)^n+2b^nc^n+(c^2)^n$

$=(ac)^n+2b^nc^n+(bd)^n$ [Using (1) and (2)]

$=a^nc^n+b^nc^n+b^nc^n+b^nd^n$

$=a^nc^n+b^nc^n+a^nd^n+b^nd^n$ [Using (3)]

$=c^n(a^n+b^n)+d^n(a^n+b^n)$

$=(a^n+b^n)(c^n+d^n)=$ R.H.S

$\therefore (b^n+c^n)^2=(a^n+b^n)(c^n+d^n)$

Thus, (a^n+b^n), (b^n+c^n), and (c^n+d^n) are in G.P.

Question 18:

If a and b are the roots of $x^2-3x+p=0$ and c,d are roots of $x^2-12x+q=0$, where a,b,c,d form a G.P. Prove that $(q+p):(q-p)=17:15$.

Solution 18:

It is given that a and b are the roots of $x^2 - 3x + p = 0$

$\therefore a + b = 3$ and $ab = p$ ....(1)

Also, c and d are the roots of $x^2 - 12x + q = 0$

$\therefore c + d = 12$ and $cd = q$ ......(2)

It is given that a, b, c, d are in G.P.

Let $a = x,\ b = xr,\ c = xr^2,\ d = xr^3$

From (1) and (2),

We obtain $x + xr = 3 \Rightarrow x(1+r) = 3$

$xr^2 + xr^3 = 12$

$\Rightarrow xr^2(1+r) = 12$

On dividing, we obtain

$$\frac{xr^2(1+r)}{x(1+r)} = \frac{12}{3}$$

$\Rightarrow r^2 = 4$

$\Rightarrow r = \pm 2$

When $r = 2,\ x = \frac{3}{1+2} = \frac{3}{3} = 1$

When $r = -2,\ x = \frac{3}{1-2} = \frac{3}{-1} = -3$

Case I: When $r = 2$ and $x = 1$, $ab = x^2 r = 2$, $cd = x^2 r^5 = 32$

$$\therefore \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$

i.e., $(q+p):(q-p) = 17:15$

Case II:

When $r = -2,\ x = -3,\ ab = x^2 r = -18,\ cd = x^2 r^5 = -288$

$$\therefore \frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$$

i.e., $(q+p):(q-p) = 17:15$

Thus, in both the cases, we obtain $(q+p):(q-p) = 17:15$.

Question 19:

The ratio of the A.M and G.M. of two positive numbers a and b, is $m:n$. Show that $a:b = \left(m + \sqrt{m^2 - n^2}\right):\left(m - \sqrt{m^2 - n^2}\right)$

Solution 19:

Let the two numbers be a and b.

A.M $=\frac{a+b}{2}$ and G.M. $=\sqrt{ab}$

According to the given condition,

$$\frac{a+b}{2\sqrt{ab}}=\frac{m}{n}$$

$$\Rightarrow \frac{(a+b)^2}{4(ab)}=\frac{m^2}{n^2}$$

$$\Rightarrow (a+b)^2=\frac{4abm^2}{n^2}$$

$$\Rightarrow (a+b)=\frac{2\sqrt{ab}m}{n} \qquad (1)$$

Using this in the identity $(a-b)^2=(a+b)^2-4ab,$ we obtain

$$(a-b)^2=\frac{4abm^2}{n^2}-4ab=\frac{4ab(m^2-n^2)}{n^2}$$

$$\Rightarrow (a-b)=\frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n} \qquad ....(2)$$

Adding (1) and (2), we obtain

$$2a=\frac{2\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$\Rightarrow a=\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

Substituting the value of a in (1), we obtain

$$b=\frac{2\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$=\frac{\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\sqrt{m^2-n^2}$$

$$=\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)$$

$$\therefore a:b=\frac{a}{b}=\frac{\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)}{\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)}=\frac{\left(m+\sqrt{m^2-n^2}\right)}{\left(m-\sqrt{m^2-n^2}\right)}$$

Thus, $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$

Question 20:

If a,b,c are in A.P; b,c,d are in G.P. and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. prove that a,c,e are in G.P.

Solution 20:

It is given that a, b, c are in A.P.

$\therefore b-a=c-b \qquad ......(1)$

It is given that b, c, d are in G.P.

$\therefore c^2=bd \qquad ......(2)$

Also, $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P.

$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$

$\frac{2}{d}=\frac{1}{c}+\frac{1}{e} \qquad ......(3)$

It has to be proved that a, c, e are in G.P. i.e., $c^2=ae$

From (1), we obtain

$2b=a+c$

$\Rightarrow b=\frac{a+c}{2}$

From (2), we obtain

$d=\frac{c^2}{b}$

Substituting these values in (3), we obtain

$\frac{2b}{c^2}=\frac{1}{c}+\frac{1}{e}$

$\Rightarrow \frac{2(a+c)}{2c^2}=\frac{1}{c}+\frac{1}{e}$

$\Rightarrow \frac{a+c}{c^2}=\frac{e+c}{ce}$

$\Rightarrow \frac{a+c}{c}=\frac{e+c}{e}$

$\Rightarrow (a+c)e=(e+c)c$

$\Rightarrow ae+ce=ec+c^2$

$\Rightarrow c^2=ae$

Thus, a, c and e are in G.P.

Question 21:

Find the sum of the following series up to n terms:

(i) $5+55+555+....$ (ii) $.6+.66+.666+...$

Solution 21:

(i) $5+55+555+....$

Let $S_n=5+55+555+....\text{to } n$ terms

$=\frac{5}{9}[9+99+999+\ldots\ldots\text{to } n \text{ terms}]$

$=\frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\ldots\text{to } n \text{ terms}\right]$

$=\frac{5}{9}\left[(10+10^2+10^3+\text{to } n \text{ terms})-(1+1+\ldots\text{to } n \text{ terms})\right]$

$=\frac{5}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$

$=\frac{5}{9}\left[\frac{10(10^n-1)}{9}-n\right]$

$=\frac{50}{81}(10^n-1)-\frac{5n}{9}$

(ii) $.6+.66+.666+\ldots$

Let $S_n=06.+0.66+0.666+\ldots.\text{to } n \text{ terms}$

$=6[0.1+0.11+0.111+\ldots.\text{to } n \text{ terms}]$

$=\frac{6}{9}[0.9+0.99+0.999+\ldots.\text{to } n \text{ terms}]$

$=\frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots.\text{to } n \text{ terms}\right]$

$=\frac{2}{3}\left[(1+1+\ldots\ n \text{ terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\ldots n \text{ terms}\right)\right]$

$=\frac{2}{3}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$

$=\frac{2}{3}n-\frac{2}{30}\times\frac{10}{9}(1-10^{-n})$

$=\frac{2}{3}n-\frac{2}{27}(1-10^{-n})$

Question 22:

Find the 20th term of the series $2\times4+4\times6+6\times8+\ldots..+n$ terms.

Solution 22:

The given series is $2\times4+4\times6+6\times8+\ldots..n$ terms

$\therefore n^{th}\text{ term}=a_n=2n\times(2n+2)=4n^2+4n$

$a_{20}=4(20)^2+4(20)=4(400)+80=1600+80=1680$

Thus, the 20th term of the series is 1680.

Question 23:

Find the sum of the first n terms of the series: $3+7+13+21+31+.....$

Solution 23:

The given series is $3+7+13+21+31+.....$

$$S=3+7+13+21+31+....+a_{n-1}+a_n$$

$$S=3+7+13+21+....+a_{n-2}+a_{n-1}+a_n$$

On subtracting both the equations, we obtain

$$S-S=\left[3+\left(7+13+21+31+...+a_{n-1}+a_n\right)+\right]-\left[\left(3+7+13+21+31+....+a_{n-1}\right)+a_n\right]$$

$$S-S=3+\left[\left(7-3\right)+\left(13-7\right)+\left(21-13\right)+....+\left(a_n-a_{n-1}\right)\right]-a_n$$

$$0=3+\left[4+6+8+....\left(n-1\right)\text{terms}\right]-a_n$$

$$a_n=3+\left[4+6+8+.....\left(n-1\right)\text{terms}\right]$$

$$\Rightarrow a_n=3+\left(\frac{n-1}{2}\right)\left[2\times4+\left(n-1-1\right)2\right]$$

$$=3+\left(\frac{n-1}{2}\right)\left[8+\left(n-2\right)2\right]$$

$$=3+\frac{\left(n-1\right)}{2}\left(2n+4\right)$$

$$=3+\left(n-1\right)\left(n+2\right)$$

$$=3+\left(n^2+n-2\right)$$

$$=n^2+n+1$$

$$\therefore\ \sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=\frac{n\left(n+1\right)\left(2n+1\right)}{6}+\frac{n\left(n+1\right)}{2}+n$$

$$=n\left[\frac{\left(n+1\right)\left(2n+1\right)+3\left(n+1\right)+6}{6}\right]$$

$$=n\left[\frac{2n^2+3n+1+3n+3+6}{6}\right]$$

$$=n\left[\frac{2n^2+6n+10}{6}\right]$$

$$=\frac{n}{3}\left[n^2+3n+5\right]$$

Question 24:

If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1+8S_1)$.

Solution 24:

From the given information,

$$S_1 = \frac{n(n+1)}{2}$$

$$S_3 = \frac{n^2(n+1)^2}{4}$$

Here, $S_3(1+8S_1) = \frac{n^2(n+1)^2}{4}\left[1+\frac{8n(n+1)}{2}\right]$

$$= \frac{n^2(n+1)^2}{4}\left[1+4n^2+4n\right]$$

$$= \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad \text{......}(1)$$

Also, $9S_2^2 = 9\frac{\left[n(n+1)(2n+1)\right]^2}{(6)^2}$

$$= \frac{9}{36}\left[n(n+1)(2n+1)\right]^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad \text{......}(2)$$

Thus, from (1) and (2), we obtain $9S_2^2 = S_3(1+8S_1)$.

Question 25:

Find the sum of the following series up to n terms:

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....$$

Solution 25:

The n^{th} term of the given series is $\frac{1^3+2^3+3^3+.....+n^3}{1+3+5+....+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+.....+(2n-1)}$

Here, $1,3,5.....(2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms as n

$$\therefore 1+3+5+....+(2n-1)=\frac{n}{2}\left[2\times1+(n-1)2\right]=n^2$$

$$\therefore a_n=\frac{n^2(n+1)^2}{4n^2}=\frac{(n+1)^2}{4}=\frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}$$

$$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\left(\frac{1}{4}K^2+\frac{1}{2}K+\frac{1}{4}\right)$$

$$=\frac{1}{4}\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\frac{n(n+1)}{2}+\frac{1}{4}n$$

$$=\frac{n\left[(n+1)(2n+1)+6(n+1)+6\right]}{24}$$

$$=\frac{n\left[2n^2+3n+1+6n+6+6\right]}{24}$$

$$=\frac{n\left(2n^2+9n+13\right)}{24}$$

Question 26:

Show that $\dfrac{1\times2^2+2\times3^2+....+n\times(n+1)^2}{1^2\times2+2^2\times3+.....+n^2\times(n+1)}=\dfrac{3n+5}{3n+1}$

Solution 26:

n^{th} term of the numerator $=n(n+1)^2=n^3+2n^2+n$

n^{th} term of the denominator $=n^2(n+1)=n^3+n^2$

$$\frac{1\times2^2+2\times3^2+....+n\times(n+1)^2}{1^2+2+2^2\times3+....+n^2\times(n+1)}=\frac{\sum_{k=1}^{n}a_K}{\sum_{k=1}^{n}a_K}=\frac{\sum_{k=1}^{n}\left(K^3+2K^2+K\right)}{\sum_{k=1}^{n}\left(K^3+K^2\right)} \qquad \text{......... (1)}$$

Here, $\sum_{k=1}^{n}\left(K^3+2K^2+K\right)$

$$=\frac{n^2(n+1)^2}{4}+\frac{2n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+11n+10\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+5)}{12} \qquad ......(2)$$

Also, $\sum_{K=1}^{n}\left(K^3+K^2\right)=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+7n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+1(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+1)}{12} \qquad ......(3)$$

From (1), (2) and (3), we obtain

$$\frac{1\times 2^2+2\times 3^2+.....+n\times(n+1)^2}{1^2\times 2+2^2\times 3+....+n^2(n+1)}=\frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}=\frac{3n+5}{3n+1}$$

Thus, the given result is proved.

Question 27:

A farmer buys a used tractor for Rs. 12000. He pays Rs. 6000 cash and agrees to pay the balance in annual installments of Rs. 500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?

Solution 27:

It is given farmer pays Rs. 6000 in cash.

Therefore, unpaid amount = Rs. 12000 – Rs. 6000 = Rs. 6000

According to the given condition, the interest paid annually is

12% of 6000, 12% of 5500, 12% of 5000 ….. 12% of 500

Thus, total interest to be paid

= 12% of 6000 + 12% of 5500 + 12% of 5000 + ….. + 12% of 500
= 12% of (6000 + 5500 + 5000 + …. + 500)
= 12% of (500 + 1000 + 1500 + ….. + 6000)
Now, the series 500, 1000, 1500 …. 6000 is an A.P. with both the first term and common difference equal to 500.
Let the number of terms of the A.P. be n.
$\therefore 6000=500+(n-1)500$
$\Rightarrow 1+(n-1)=12$
$\Rightarrow n=12$
$\therefore$ Sum of the A.P
$=\frac{12}{2}\left[2(500)+(12-1)(500)\right]=6[1000+5500]=6(6500)=39000$
Thus, total interest to be paid
= 12% of (500 + 1000 + 1500 + ….. + 6000)
= 12% of 39000 = Rs. 4680
Thus, cost of tractor = (Rs. 12000 + Rs. 4680) = Rs. 16680.

Question 28:
Shamshad Ali buys a scooter for Rs. 22000. He pays Rs. 4000 cash and agrees to pay the balance in annual installment of Rs. 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Solution 28:
It is given that Shamshad Ali buys a scooter for Rs. 22000 and pays Rs. 4000 in cash.
$\therefore$ Unpaid amount = Rs. 22000 – Rs. 4000 = Rs. 18000
According to the given condition, the interest paid annually is
10% of 18000, 10% of 17000, 10% of 16000 ….. 10% of 1000
Thus, total interest to be paid
= 10% of 18000 + 10% of 17000 + 10% of 16000 + ….. + 10% of 1000
= 10% of (18000 + 17000 + 16000 + ….. + 1000)
= 10% of (1000 + 2000 + 3000 + ….. + 18000)
Here, 1000, 2000, 3000 …. 18000 forms an A.P. with first term and common difference both equal to 1000.
Let the number of terms be n.
$\therefore 18000=1000+(n-1)(1000)$
$\Rightarrow n=18$
$\therefore 1000+2000+....+18000=\frac{18}{2}\left[2(1000)+(18-1)(1000)\right]$
$=9[2000+17000]$
$=171000$
$\therefore$ Total interest paid = 10% of (18000 + 17000 + 16000 + …. + 1000)
= 10% of Rs.171000 = Rs. 17100
$\therefore$ Cost of scooter = Rs. 22000 + Rs. 17100 = Rs. 39100.

Question 29:

A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8^{th} set of letter is mailed.

Solution 29:

The numbers of letters mailed forms a G.P.: $4, 4^2, \ldots . 4^8$

First term = 4

Common ratio = 4

Number of terms = 8

It is known that the sum of n terms of a G.P. is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$$\therefore S_8 = \frac{4\left(4^8 - 1\right)}{4-1} = \frac{4\left(65536 - 1\right)}{3} = \frac{4\left(65535\right)}{3} = 4\left(21845\right) = 87380$$

It is given that the cost to mail one letter is 50 paisa.

$$\therefore \text{ Cost of mailing 87380 letters } = \text{Rs.}87380 \times \frac{50}{100} = \text{Rs.}43690$$

Thus, the amount spent when 8^{th} set of letter is mailed is Rs. 43690.

Question 30:

A man deposited Rs. 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

Solution 30:

It is given that the man deposited Rs. 10000 in a bank at the rate of 5% simple interest annually.

$$= \frac{5}{100} \times \text{Rs.}10000 = \text{Rs.}500$$

$$\therefore \text{ Interest in first year } 10000 + \underbrace{500 + 500 + \ldots . + 500}_{\text{14 times}}$$

$\therefore$ Amount in 15^{th} year

$$= \text{Rs.}10000 + 14 \times \text{Rs.}500$$

$$= \text{Rs.}10000 + \text{Rs.}7000$$

$$= \text{Rs.}17000$$

$$\text{Amount after 20 years } = \text{Rs.}10000 + \underbrace{500 + 500 + \ldots . + 500}_{\text{20times}}$$

$$= \text{Rs.}10000 + 20 \times \text{Rs.}500$$

$$= \text{Rs.}10000 + \text{Rs.}10000$$

$$= \text{Rs.}20000.$$

Question 31:

A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Solution 31:

Cost of machine = Rs. 15625

Machine depreciates by 20% every year.

Therefore, its value after every year is 80% of the original cost i.e., $\frac{4}{5}$ of the original cost.

$\therefore$ Value at the end of 5 years $=15625\times\underbrace{\frac{4}{5}\times\frac{4}{5}\times....\times\frac{4}{5}}_{5\text{ times}}=5\times1024=5120$

Thus, the value of the machine at the end of 5 years is Rs. 5120.

Question 32:

150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution 32:

Let x be the number of days in which 150 workers finish the work.

According to the given information,

$150x=150+146+142+....(x+8)\text{terms}$

The series $150+146+142+....(x+8)\text{terms}$ is an A.P. with first term 146, common difference -4 and number of terms as $(x+8)$

$\Rightarrow 150x=\frac{(x+8)}{2}\left[2(150)+(x+8-1)(-4)\right]$

$\Rightarrow 150x=(x+8)\left[150+(x+7)(-2)\right]$

$\Rightarrow 150x=(x+8)(150-2x-14)$

$\Rightarrow 150x=(x+8)(136-2x)$

$\Rightarrow 75x=(x+8)(68-x)$

$\Rightarrow 75x=68x-x^2+544-8x$

$\Rightarrow x^2+75x-60x-544=0$

$\Rightarrow x^2+15x-544=0$

$\Rightarrow x^2+32x-17x-544=0$

$\Rightarrow x(x+32)-17(x+32)=0$

$\Rightarrow (x-17)(x+32)=0$

$\Rightarrow x=17$ or $x=-32$

However, x cannot be negative.

$\therefore x = 17$

Therefore, originally, the number of days in which the work was completed is 17. Thus, required number of days $= (17+8) = 25$.